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COMMONWEALTH SCIENTIFIC AND INDUSTRIAL RESEARCH ORGANIZATION

DIVISION of FISHERIES and OCEANOGRAPHY

Report No. 66

ON THE CONCEPT OF FISH MORTALITY RATES IN THE EXPLOITED PHASE AND THEIR ESTIMATION BY SAMPLING THE COMMERCIAL CATCH, WITH SPECIAL ATTENTION TO ROCK LOBSTER FISHERIES

By W. J. Stamper

Marine Laboratory
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CONTENTS

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ABSTRACT

The catchability coefficient q and natural mortality coefficient M of a cohort in the exploited phase are defined as the expected values, given the set of recruits, of the arithmetic means of the corresponding coefficients of those members of the cohort that are still alive. It is shown that it would be useful, in population modelling, to know values of q and M defined in this manner.

A general method is developed for the estimation of q and M, of assumed functional form, by sampling the commercial catch. The method does not involve an intermediate step of computing total mortality rates, which do not exist when the effort is not a differentiable function of time.

Special attention is given to the case of rock lobster fisheries. Real fisheries data are used only for illustration.

I. INTRODUCTION

Gulland (1969) describes a method whereby the catchability coefficient, q, and natural mortality coefficient, M, are estimated from catch and effort data, but only as constants. Moreover, the formulation of that method imposes the restriction of having the total mortality coefficient, Z, existing (i.e. effort a differentiable function of time) and constant for a period, for which it is estimated, and then changing to a new level. This restriction is imposed also by a method due to Murphy (1965) whereby the fishing mortality coefficient, F, for a cohort is estimated from catch data when M is known.

The following general method incorporates the improvement of allowing q and M to have any specified functional form, q = constant and M = constant (or known) being only one particular case. Also, the above restriction is removed by omitting the unnecessary intermediate step of calculating Z values.

q and M are defined for a cohort in the exploited phase in terms of corresponding parameters of the member individuals. Before developing the general model for estimating q and M in the exploited phase by sampling the commercial catch, it is shown that it would be useful, in population modelling, to know the values of cohort mortality rates defined in this manner.

Special attention is given to the case for rock lobster fisheries. Real fisheries data are used only for illustration in the general theme and should not be taken as the most accurate data available. It is hoped that a supplement to this work will eventually be prepared, wherein the model will be applied to data on the Australian southern rock lobster, *Jasus novaehollandiae* Holthuis, to estimate the q values and the M values for that species.

Notations adopted in this paper are outlined in Appendix I.

II. THE CONCEPT OF MORTALITY RATES IN THE EXPLOITED PHASE

Partition the entire time axis into intervals and call them "cohort" intervals. On each cohort interval choose a point and call it the "birthdate" of that interval.

For a certain type of fish and this partitioning of time, assume the existence of a definite "recruitment cohort age" X_r , large enough such that the birthdates of all fish born on any cohort interval are before the date $X = X_r$, where X denotes the age of the interval, and small enough such that each of these fish is too young to be caught when $X = X_r$.

Let R be the entire set of this type of fish each of which is alive when its cohort interval (the interval on which it was born) is recruited ($X = X_r$) . Thus R is the set of recruits over all time. Call a "cohort" each subset of those members of R that have the same cohort interval, and perhaps also some other common characteristics e.g. same sex, inhabit same area from recruitment onwards (called a "zone"). At any time t some members of R are not yet born, some are alive and some are dead; call the "standing crop" S(t) those fish that are alive in the cohorts with $X \geq X_r$.

Call the "exploited phase" of a cohort that period of time after the date $X = X_r$. This modelling concerns stochastic death processes incurred by cohorts in the exploited phase.

Let g(t) be the fishing effort (in a zone) on R from the beginning of the fishery up to time t . g(t) is a non-decreasing continuous function of t , but in general not everywhere differentiable. During (t,t+dt) let the fishing effort be dg . Consider a definite fish ε R which at t is alive in the exploited phase and has size h and age x . During (t,t+dt) it might be caught, might die of natural causes (i.e. die but not be caught), or might remain alive. The probabilities (designated by "Pr(...)") associated with these events are defined as follows:

Consider the entire set of situations, past, present and future, of which the above situation is a particular instance, all with certain common attributes, namely where a fish h,x, ε R , in the exploited phase, is subjected to dg during a time interval of duration dt with fishing gear in certain condition (e.g. mesh size, type of bait) when certain types and quantities of natural foods are available, when a certain pollution level and a certain predation level exists, etc.. Then

Pr(caught) $\equiv$ fraction of situations when the fish is caught during the relevent time interval of duration dt ,

Pr(natural death) $\equiv$ fraction of situations when the fish dies naturally,

Pr(remains alive) $\equiv$ fraction of situations when the fish remains alive.

These events are mutually exclusive and exhaustive, so Pr(dies) = Pr(caught) + Pr(natural death) . If dt is small enough, Pr(caught) will not depend upon the magnitude of Pr(natural death) , i.e. upon dt, and Pr(natural death) will not depend upon dg . So, if during (t,t+dt) there are no sudden changes in fishing methods, natural foods, pollution level, predation level, etc., then Pr(caught) is proportional to dg, and Pr(natural death) is proportional to dt . The proportionality constants will in general depend on h, x and t (also on sex and zone - for instance, see Section XIII). So we can write Pr(caught) = $q(h,x,t)dg$ and Pr(natural death) = $M(h,x,t)dt$.

It is not feasible to estimate $q(h,x,t)$, $M(h,x,t)$ by sampling the catch, because the size of a fish caught is not known sufficiently accurately at any previous time. However, it is possible to trace the progress of a cohort, so cohort averages q and M (defined below) can be estimated.

Consider now the exploited phase of a cohort $\subset R$, which has N_o fish and is recruited at t_o . Let q_i, M_i be respectively the functions $q(h,x,t)$, $M(h,x,t)$ for the i^{th} fish of this cohort, and let $h_i(x)$ be its size at age x . Assume $h_i(x)$ to be a deterministic function of x (but possibly different for different i), making q_i and M_i deterministic functions of t . (Note that in practice growth is usually stochastic, i.e. $h_i(x)$ is a variate, making q_i and M_i variates. However, in this paper these variations in the $h_i(x)$ are ignored except in the calculation of the expected value and the variance of the cohort/size key (Case III in Fig.2).)

Let $t \geq t_o$. In the following definitions the integrals are over the path shown in Figure 1.

$$dP_i \equiv q_i dg + M_i dt \quad . \qquad P_i \equiv \int_{Q_o} dP_i \quad .$$

$$\omega_i \equiv e^{-P_i} / \sum_1^{N_o} e^{-P_i} \quad .$$

$$d\overline{P} \equiv \sum \omega_i dP_i \quad . \qquad \overline{P} \equiv \int_{Q_o} d\overline{P} \quad .$$

$a_i \equiv (1,0)$ if individual i(alive,not alive) at t .

$$N \equiv \text{number of fish } \varepsilon \text{ cohort alive at } t \text{ , } = \sum_1^{N_o} a_i \quad .$$

$$\tilde{q} \equiv \frac{1}{N} \sum q_i a_i \quad . \qquad \tilde{M} \equiv \frac{1}{N} \sum M_i a_i \quad .$$

$$dP \equiv \frac{1}{N} \sum a_i dP_i = \tilde{q}\, dg + \tilde{M}\, dt \quad . \qquad P \equiv \int_{Q_o} dP \quad .$$

Note that the variates N, $\tilde{q}$, $\tilde{M}$, P are functions of t, and X (to specify the cohort).

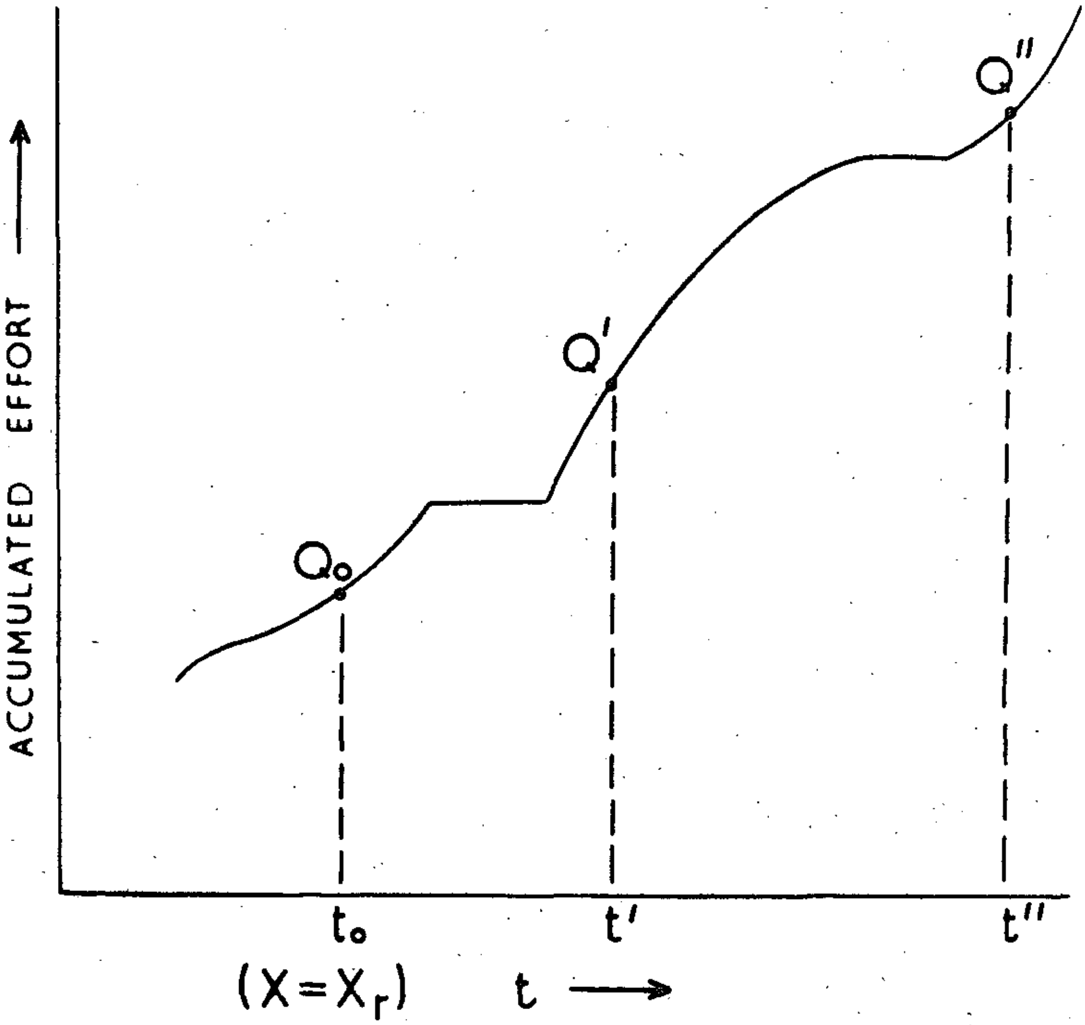

Fig. 1. Path of integration of dP_i , $d\overline{P}$, dP .

Theorem: –

(1) $E(N| = N_0 e^{-\overline{P}}$.

(2) $CV(N|, CV(\tilde{q}|, CV(\tilde{M}|, CV(P| \lesssim E^{-\frac{1}{2}}(N|$.

(3) $E(P| = \overline{P}(1+\delta)$, where $\delta \lesssim E^{-1}(N|$.

Proof: –

(1) $E(a_i| = e^{-P_i}$, $\sigma^2(a_i| = e^{-P_i}(1-e^{-P_i})$.

$\therefore E(N| = \sum e^{-P_i}$. $\therefore \dfrac{dE(N|}{E(N|} = -d\overline{P}$.

(2) $CV(N| = (1-\sum \omega_i e^{-P_i})^{\frac{1}{2}} / E^{\frac{1}{2}}(N| < E^{-\frac{1}{2}}(N|$.

Let $r \equiv \sum q_i a_i$. $\therefore \tilde{q} = r/N$.

So that the following expression for $CV(r|$ cannot equal 0/0 , assume (without loss of generality) that at recruitment (t_0), for every i, $q_i \simeq 0$ but nevertheless $q_i > 0$.

$$CV(r| = \{\sum q_i^2 e^{-P_i}(1-e^{-P_i})\}^{\frac{1}{2}}/\sum q_j e^{-P_j} < \{\sum q_i^2 e^{-P_i}\}^{\frac{1}{2}}/\sum q_j e^{-P_j}$$

$$= \left[\sum_i \left\{ \frac{q_i e^{-P_i}}{\sum_j q_j e^{-P_j}} q_i \right\} \Big/ \sum_i \left\{ \frac{e^{-P_i}}{\sum_j e^{-P_j}} q_i \right\} \right]^{\frac{1}{2}} \Big/ E^{\frac{1}{2}}(N| \sim E^{-\frac{1}{2}}(N| ,$$

since the numerator is the ratio of two weighted means of the q_i .

$\therefore CV(\tilde{q}| \lesssim E^{-\frac{1}{2}}(N|$ by Appendix II(4). Similarly $CV(\tilde{M}| \lesssim E^{-\frac{1}{2}}(N|$.

$\therefore CV(\tilde{q}|, CV(\tilde{M}| \lesssim E^{-\frac{1}{2}}(N|$ at any point on the path of integration of P, where N refers to the upper limit of integration.

$\therefore E(dP|(1-E^{-\frac{1}{2}}[N|) \lesssim dP \lesssim E(dP|(1+E^{-\frac{1}{2}}[N|)$.

$\therefore \sigma(P| \lesssim E(P|E^{-\frac{1}{2}}(N|$.

(3) By Appendix II(2),

$$E(dP| = \frac{E(\Sigma a_i dP_i|}{E(N|}(1+CV^2[N|-\rho[\Sigma a_i dP_i, N|CV[\Sigma a_i dP_i|CV[N|+O(\varepsilon^3)),$$

where $\varepsilon \equiv CV(\Sigma a_i dP_i| + CV(N|$.

Now $CV(\Sigma a_i dP_i| \lesssim E^{-\frac{1}{2}}(N|$ (cf. above examination of $CV[r|$) .

$\therefore E(dP| = (\Sigma \omega_i dP_i)(1+\delta) = d\overline{P}(1+\delta)$, where $|\delta| \lesssim E^{-1}(N|$.

Q.E.D.

$\tilde{q}$, $\tilde{M}$ can be defined by the statement that the respective expected proportions of deaths in the cohort due to fishing, natural causes in any proceeding small time interval dt are $\tilde{q}\,dg$, $\tilde{M}\,dt$. (Note that if g(t) can be replaced, to sufficient approximation, by a curve with continuous first derivative, g' , we can write dg = g'dt . Then $\tilde{q}\,dg = \tilde{F}\,dt$ where $\tilde{F} \equiv \tilde{q}g'$, and the expected total number of deaths can be written $\tilde{Z}\,dt$, where $\tilde{Z} \equiv \tilde{F} + \tilde{M}$.)

The model developed herein is for the estimation of $q \equiv E(\tilde{q}|$, $M \equiv E(\tilde{M}|$.

Knowledge of these functions is useful for population modelling. They might be used, for instance, in equations such as

$E(N| = N_0 e^{-E(P|} = N_0 e^{-\int_{Q_0} q\,dg + M dt}$, the former of which holds when $E(N| >> 1$ (by the theorem – (1),(3)).

Under the more stringent condition $E^{\frac{1}{2}}(N| >> 1$, which will be assumed to hold later in developing the model, $CV(N| << 1$ (by the theorem - (2)). In such circumstances it would be justifiable to write "$N = N_0 e^{-\int_{Q_0} q\,dg + M\,dt}$". (This is not done in this paper.) In such deterministic descriptions q , M have been called respectively the catchability coefficient, natural mortality coefficient (and $F \equiv E[\tilde{F}|$, $Z \equiv E[\tilde{Z}|$ respectively the fishing mortality coefficient, total mortality coefficient) – see, for instance, Gulland (1969).

Note that when $E^{\frac{1}{2}}(N| >> 1$, by the theorem – (2), $CV(\tilde{q}|$, $CV(\tilde{M}|$, $CV(P| << 1$ (similarly $CV[\tilde{F}|$, $CV[\tilde{Z}| << 1$). However, the population dynamics cannot strictly be called deterministic in this situation, for there are still variates, having large coefficients of variation (i.e. >> 1), which can be defined on the system – for instance the catch from the cohort taken over a small time period. It cannot even be claimed that all functions of N , $\tilde{q}$, $\tilde{M}$; P , ($\tilde{F}$, $\tilde{Z}$) have small coefficients of variation.

Note also that if the cohort interval is short enough and if q(h,X,t) , M(h,X,t) are independent of h , then $\tilde{q}$, $\tilde{M}$ are not variates.

III. FLOW SUMMARY OF GENERAL MODEL FOR ESTIMATION OF q , M BY SAMPLING THE COMMERCIAL CATCH

Figures 2 - 5 are a flow summary of the general model developed herein. Flow chart symbols are given in Figure 12.

The details of Figures 2 - 5 are elaborated in the remainder of the text.

IV. BASIC EQUATION FOR ESTIMATION OF q , M BY SAMPLING THE COMMERCIAL CATCH

Let s be the catch taken (in the zone) from R over the sampling interval $(t - \frac{\delta t}{2}, t + \frac{\delta t}{2})$, where $t - \frac{\delta t}{2} \geq t_o$. Then $s \subset \Lambda \subset R$, where $\Lambda \equiv S(t - \frac{\delta t}{2}) \cup$ (all cohorts recruited in $[t - \frac{\delta t}{2}, t + \frac{\delta t}{2}]$).

Let there be c fish ε s belonging to the cohort under consideration (i.e. the cohort of Section II, recruited at t_o). These are the members of the cohort that are taken from $S(t - \frac{\delta t}{2})$ over $(t - \frac{\delta t}{2}, t + \frac{\delta t}{2})$. It will be shown later that s is sampled to determine an estimator ξ of $E(c|\Lambda)$. ξ is assumed unbiased, i.e. $E(\xi|\Lambda) = \mu \equiv E(c|\Lambda)$.

Pr(i^{th} fish is caught during sampling interval|alive at $t - \frac{\delta t}{2}$) $= \int q_i \, dg$, where the integration is over $(t - \delta t/2 , t + \delta t/2)$. This integral can be regarded equal to $q_i \delta g$, where q_i refers to time t , and δg is the effort (in the zone) during the sampling interval. Hence

$\mu = \sum_{1}^{N_o} \bar{a}_i q_i \delta g$ where $\bar{a}_i$ is the variate a_i but at time $t - \delta t/2$.

Now $E(\Sigma \bar{a}_i q_i | / E(\Sigma a_i q_i | = \sum \Omega_i (1 + q_i \frac{\delta g}{2} + M_i \frac{\delta t}{2} + O[\delta g + \delta t]^2)$,

where $\Omega_i \equiv q_i e^{-P_i} / \sum q_j e^{-P_j}$.

Assuming $q \, \delta g + M \, \delta t << 1$, Then the weighted mean

$\sum \omega_i (q_i \delta g + M_i \delta t) << 1$, where $\omega_i = e^{-P_i} / \sum e^{-P_j}$. Then it is reasonable to assume the weighted mean

$\sum \Omega_i (q_i \delta g + M_i \delta t) << 1$.

Whence we can write

$E(\Sigma \bar{a}_i q_i | = E(\Sigma a_i q_i |$. Therefore

$\frac{E(\mu/\delta g|}{E(N|} = \sum \omega_i q_i = q(1 + \delta)$ where $|\delta| \lesssim E^{-1}(N|$.

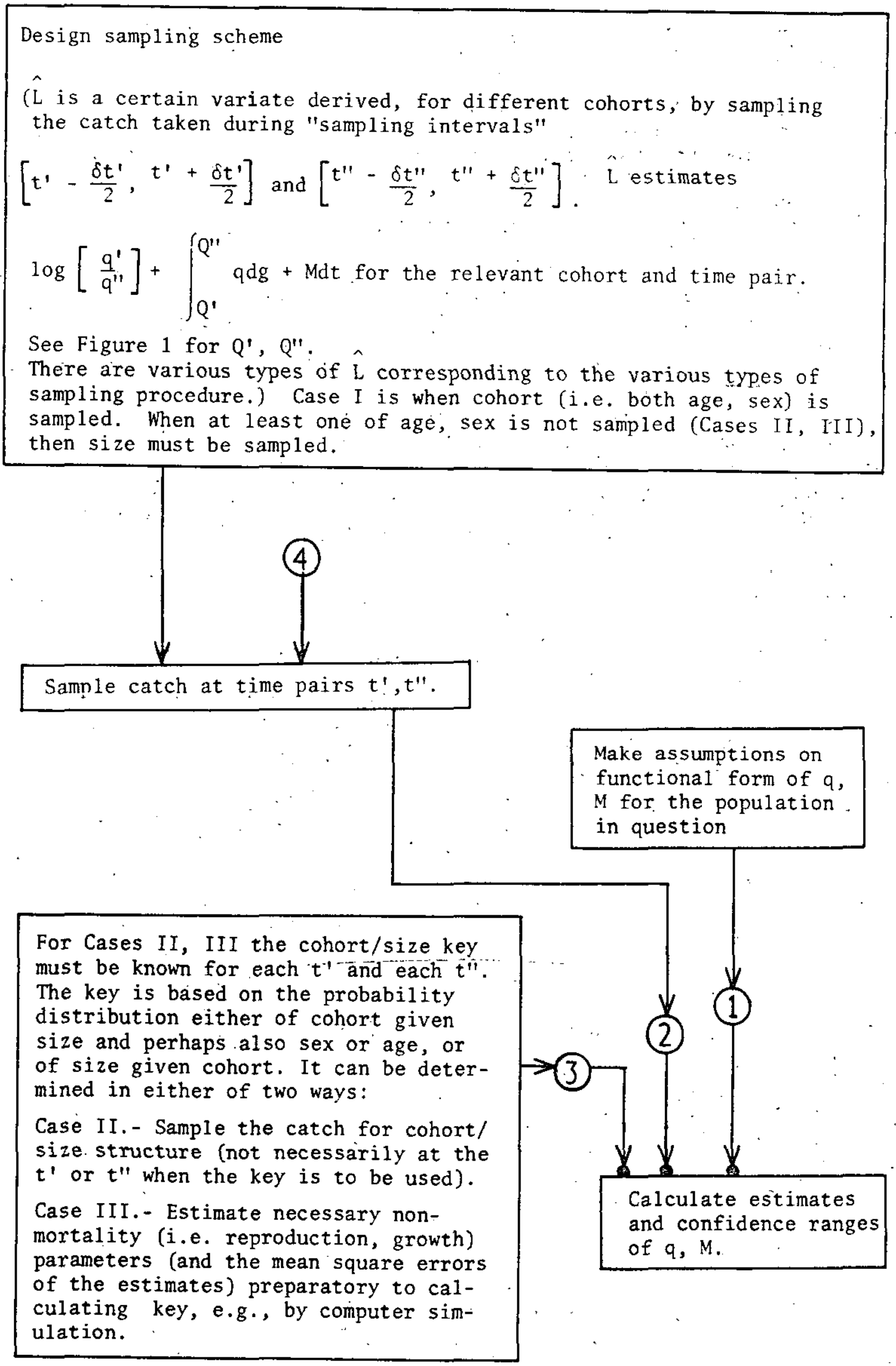

Fig. 2. Commencement of flow summary of general model for estimation of q, M by sampling the commercial catch. Flow is completed in Figures 3 - 5.

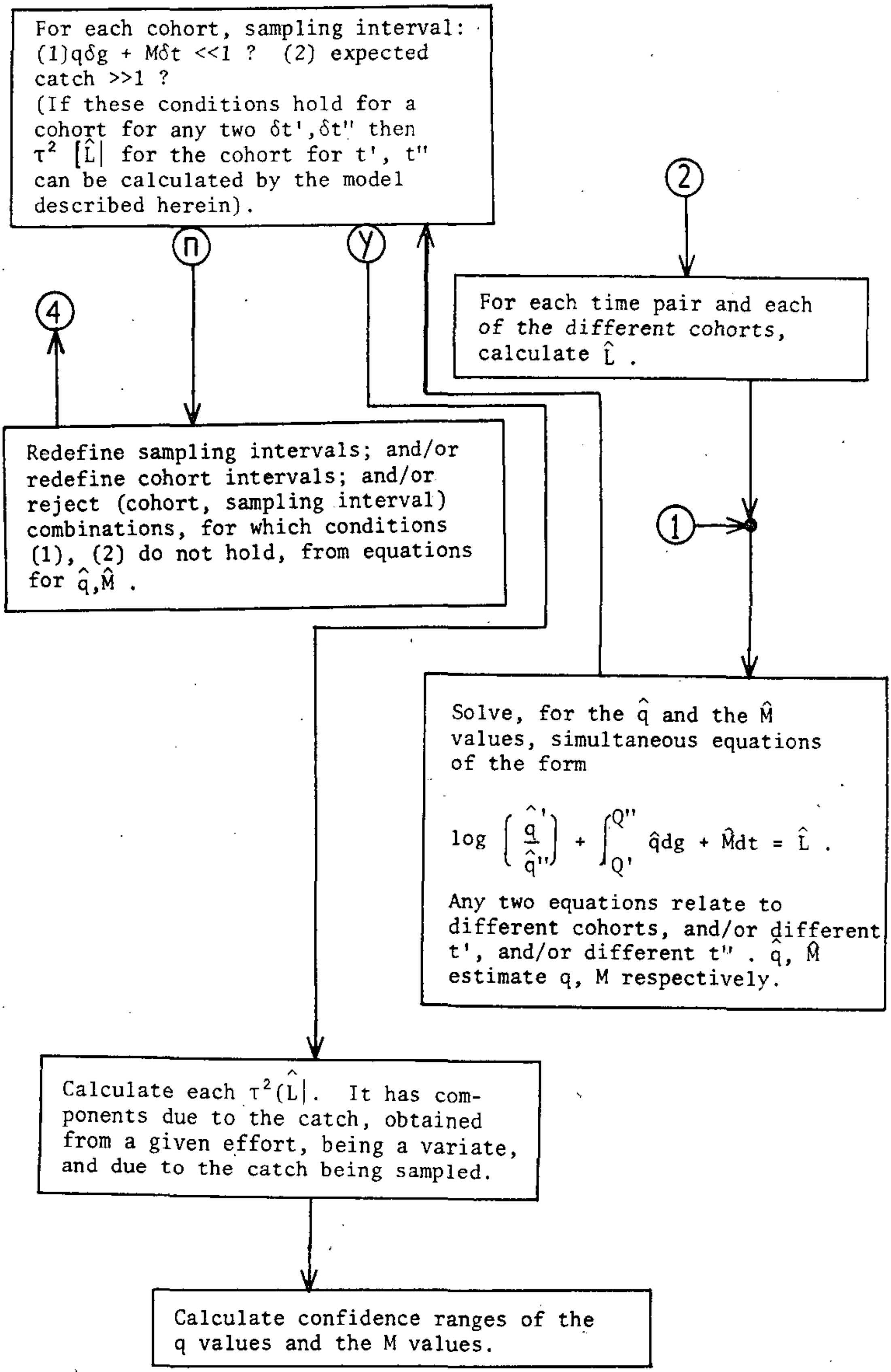

Fig. 3. Flow summary for calculation of estimates and confidence ranges of q, M *for Case I. Continues from Figure 2.*

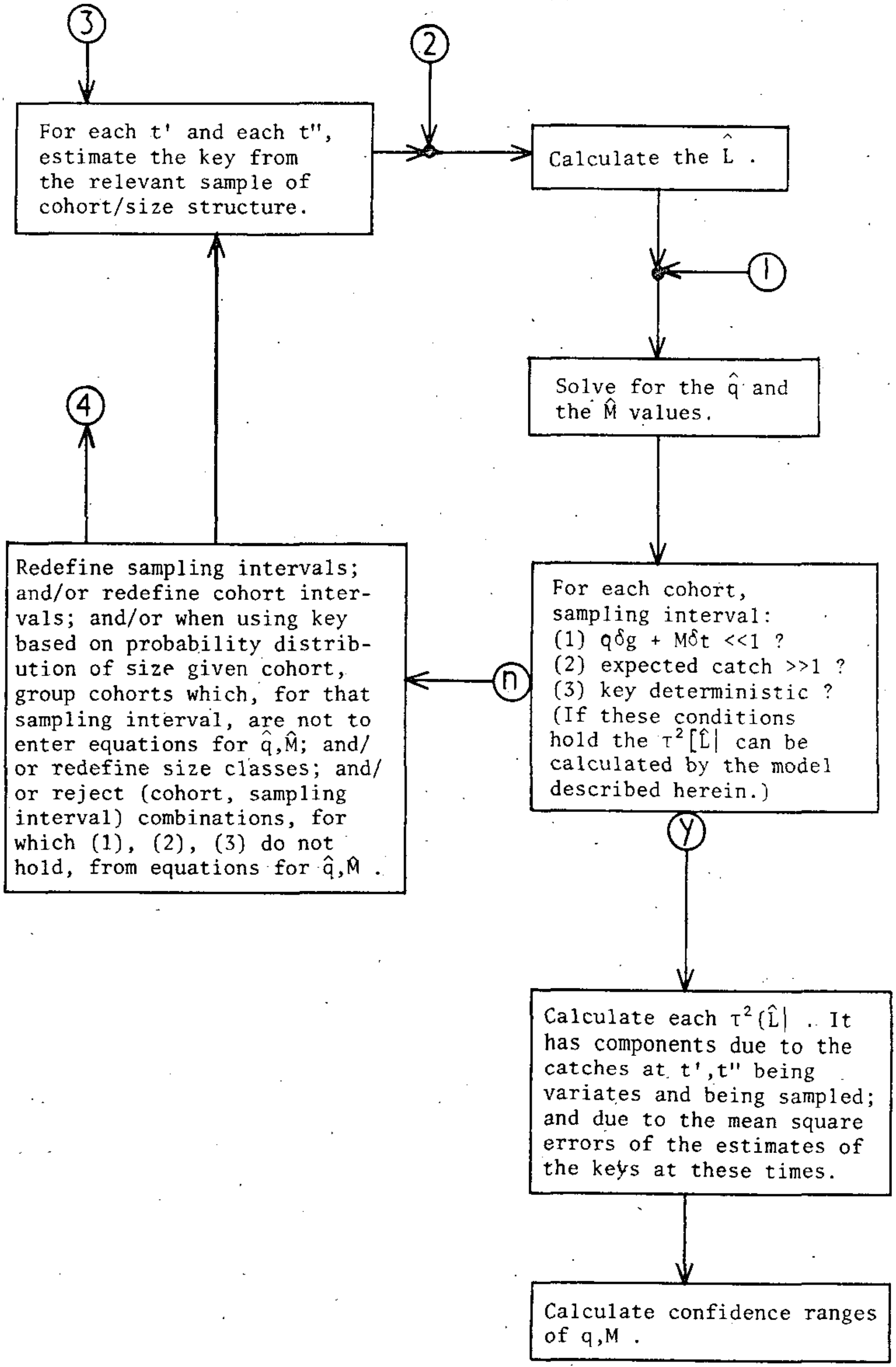

Fig. 4. Flow summary for calculation of estimates and confidence ranges of q , M for Case II. Continues from Figure 2.

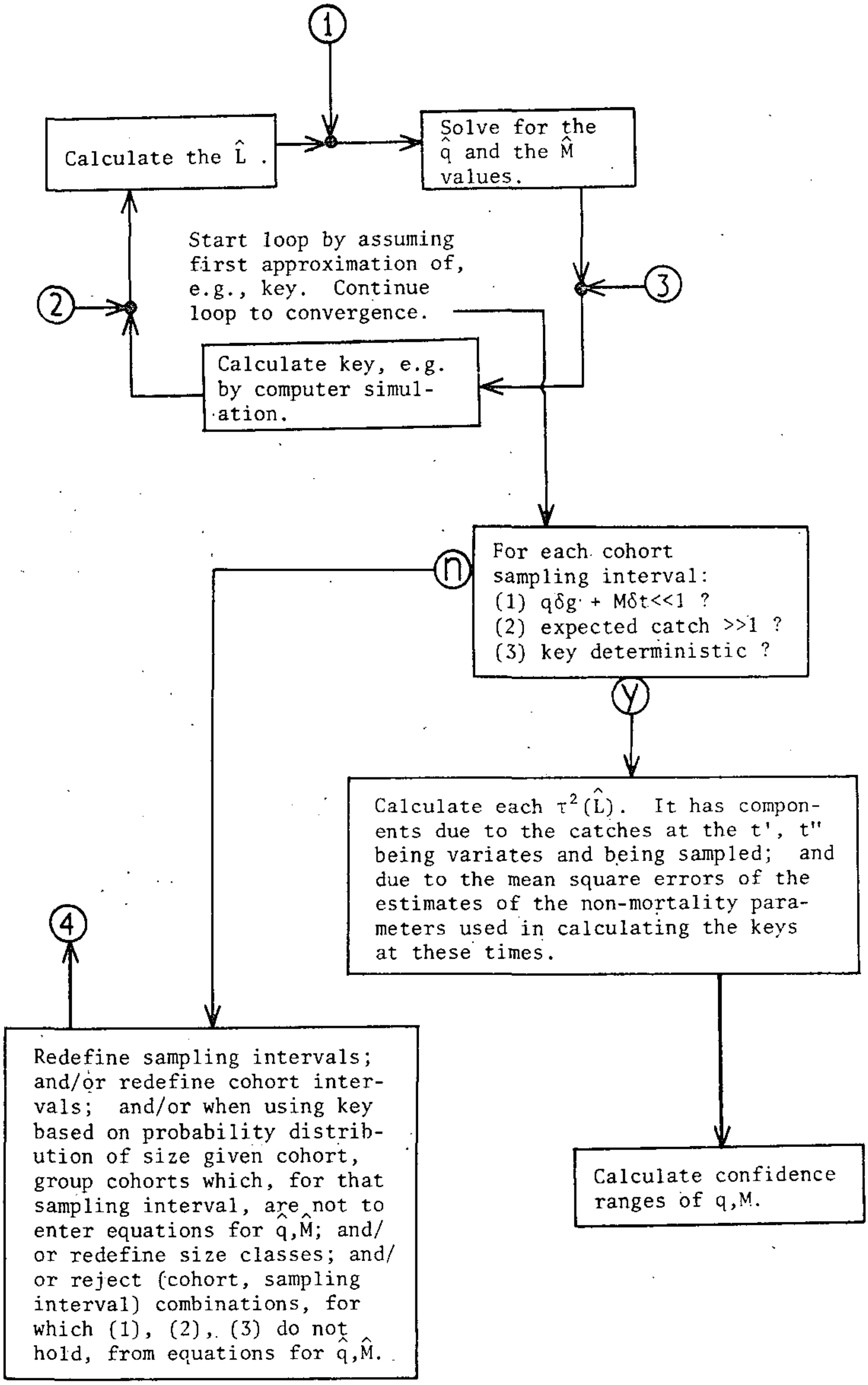

Fig. 5. Flow summary for calculation of estimates and confidence ranges of q , M for Case III. Continues from Figure 2.

Let $t' - \delta t'/2 \geq t_o$, $t'' - \delta t''/2 \geq t_o$, $t'' > t'$ (see Figure 1). Let $\delta g'$, s' , μ' , ξ' and $\delta g''$, s'' , μ'' , ξ'' refer respectively to the sampling intervals $(t' - \delta t'/2 , t' + \delta t'/2)$ and $(t'' - \delta t''/2 , t'' + \delta t''/2)$. Let N', q' , $\bar{P}'$, and N'', q'' , $\bar{P}''$ refer respectively to t' and t'' . Then, by the theorem of Section II – (1),

$$\frac{E(\xi'/\delta g'|}{E(\xi''/\delta g''|} = \frac{E(\mu'/\delta g'|/E(N'|}{E(\mu''/\delta g''|/E(N''|} \cdot \frac{N_o e^{-\bar{P}'}}{N_o e^{-\bar{P}''}}$$

$$= e^{\bar{P}''-\bar{P}'} \frac{q'}{q''}(1+\varepsilon) \quad , \text{ where } |\varepsilon| \lesssim E^{-1}(N''| \; .$$

$$\therefore \log \left\{\frac{E(\xi'/\delta g'|}{E(\xi''/\delta g''|}\right\} = \log \left\{\frac{q'}{q''}\right\} + \bar{P}'' - \bar{P}' + \varepsilon$$

$$= \log \left\{\frac{q'}{q''}\right\} + \bar{P}'' - \bar{P}' \text{ when } \bar{P}'' - \bar{P}' >> E^{-1}(N''|$$

; it will now be shown that this is the case provided both the conditions $E(N''| >> 1$ and $E(N' - N''| >> 1$ hold:

For let $\Theta \equiv (\bar{P}'' - \bar{P}')E(N''|$, $\alpha \equiv E(N'|/E(N''|$. Then, by the theorem of Section II - (1), $\Theta = E(N''| \log \alpha$

$> E(N''|$ when $\alpha > e$. Also,

$$\frac{\Theta}{E(N' - N''|} = \frac{\log \alpha}{\alpha - 1}$$

~ 1 when $1 \leq \alpha \leq e$. Q.E.D.

Let $L \equiv \log\left(\frac{q'}{q''}\right) + \int_{Q'}^{Q''} q\,dg + M\,dt$ (see Figure 1). Then, by the above and the theorem of Section II - (3) , when $E(N''| >> 1$ and $E(N' - N''| >> 1$,

$$L = \log\left\{\frac{E(\xi'/\delta g'|}{E(\xi''/\delta g''|}\right\} \; .$$

This is the basic equation for the estimation of q , M by sampling the commercial catch. The right hand side is replaced by the estimator $\hat{L}$ developed in the following pages, and the unknown functions q , M in the left hand side are replaced by their estimators $\hat{q}$, $\hat{M}$. A set of such equations is then solved for the $\hat{q}$, $\hat{M}$ values. (Note: the estimators of q' , q'' are denoted by $\hat{q}'$, $\hat{q}''$ respectively.)

V. THE ESTIMATOR ξ

As explained in Section IV, s is sampled to determine an estimator ξ such that $E(\xi|\Lambda) = \mu$.

Consider the cohorts that can be represented in s (given Λ) to be grouped, since when formulating the inverse cohort / size key (see Section V (b)) it might be necessary to combine some of these cohorts. However, ξ is determined for single cohorts only, so such a cohort is not grouped with any other. Hence most "cohort groups" will contain only one cohort. Each fish ε s will have a size-class H when caught and will belong to a definite cohort group Γ ; let $c_{\alpha\beta}$ be the number with $H = H_\alpha$, $\Gamma = \Gamma_\beta$. Hence c is one of the $c_{\cdot\beta}$, say $c_{\cdot b}$.

There are various types of ξ corresponding to various types of sampling proceedure.

The factors determining Γ of a fish ε s are its cohort age X at t and (usually) its sex S (=1,2 if male,female). Since age, sex and size are usually correlated, if one of these factors is hard to determine then size is sampled and either a smaller sample of the factor is taken, or it is not sampled.

Let $X_{\beta(j)}$ be the age at t of $\beta(j)$, the j^{th} cohort $\subset \Gamma_\beta$. When all cohorts $\subseteq \Gamma_\beta$ are of the same sex S_β , then $\Gamma = \Gamma_\beta$ is equivalent to X (at t) ε $X_{\beta(*)}$, $S = S_\beta$ .Hence, under these circumstances, the subscript β can be replaced by the subscript pair $[X_{\beta(*)}, S_\beta]$. When Γ_β consists of a single cohort, $X_{\beta(*)}$ can be written as X_β .

(a) *Case I*

For Case I, s is sampled for cohort, i.e. for both age and sex. Hence ξ estimates c , and the cohort / size key is not used.

(i) *Determine Cohort of Each Fish in Sample from* s

Example: — A random sample of n fish is taken from s . In the sample, let there be y_b fish ε Γ_b . Let $\xi \equiv c_{\cdot\cdot} y_b / n$. ∴

$$E(\xi|s) = c \ , \quad \sigma^2(\xi|s) = \frac{c(c_{\cdot\cdot} - n)(c_{\cdot\cdot} - c)}{n(c_{\cdot\cdot} - 1)} \ .$$

(ii) *Determine Size and Perhaps One of Age, Sex of Each Fish in Sample from* s . *In Smaller Sample Determine Size and Cohort of Each*

Example: — Suppose that the age of a certain mollusc can be easily determined by counting growth rings on the shell but that the sex can be determined only by dissection. A random sample s_1 of n fish is taken from s , and the size and age of each is measured. Let $Y_{\alpha\beta}$ be the frequency in s_1 of $H = H_\alpha$, $\Gamma = \Gamma_\beta$.

A random sample is taken from those members of s_1 that have $X(\text{at } t) = X_b$, and the sex determined for each. In this latter sample let $y_{\alpha b}$ be the frequency of $H = H_\alpha$, $\Gamma = \Gamma_b$. Sampling is continued past a predetermined level till $y_{\alpha[X_b,\cdot]} > 0$ for every α with $Y_{\alpha[X_b,\cdot]} > 0$.

$$\text{Let } \xi \equiv \sum_{\substack{\alpha \\ (Y_{\alpha[X_b,\cdot]}>0)}} \frac{c_{\cdot\cdot}}{n} Y_{\alpha[X_b,\cdot]} \frac{y_{\alpha b}}{y_{\alpha[X_b,\cdot]}} . \quad \therefore$$

$$E(y_{\alpha b}/y_{\alpha[X_b,\cdot]} \mid s_1) = Y_{\alpha b}/Y_{\alpha[X_b,\cdot]} . \quad \therefore E(\xi|s) = E\left(\frac{c_{\cdot\cdot}}{n} Y_{\cdot b} | s\right) = c .$$

$$\sigma^2(\xi|s) = \sigma^2\{E(\xi|s_1)|s\} + E\{\sigma^2(\xi|s_1)|s\} .$$

$$\sigma^2\{E(\xi|s_1)|s\} = \sigma^2\left(\frac{c_{\cdot\cdot}}{n} Y_{\cdot b}|s\right) = \frac{c(c_{\cdot\cdot}-n)(c_{\cdot\cdot}-c)}{n(c_{\cdot\cdot}-1)} .$$

$$\sigma^2(\xi|s_1) = \sigma^2\left(\sum_{\substack{\alpha \\ (Y_{\alpha[X_b,\cdot]}>0)}} \frac{c_{\cdot\cdot}}{n} Y_{\alpha[X_b,\cdot]} \frac{y_{\alpha b}}{y_{\alpha[X_b,\cdot]}} \Big| s_1\right) .$$

For any α', α'' such that $\alpha' \neq \alpha''$,

$$E\left\{(y_{\alpha' b}/y_{\alpha'[X_b,\cdot]})(y_{\alpha'' b}/y_{\alpha''[X_b,\cdot]})|s_1\right\} =$$

$$E\left[E\left\{(y_{\alpha' b}/y_{\alpha'[X_b,\cdot]})(y_{\alpha'' b}/y_{\alpha''[X_b,\cdot]})|y_{\alpha'[X_b,\cdot]}, y_{\alpha''[X_b,\cdot]}\right\}|s_1\right] =$$

$$E\left[E(y_{\alpha' b}/y_{\alpha'[X_b,\cdot]}|y_{\alpha'[X_b,\cdot]})E(y_{\alpha'' b}/y_{\alpha''[X_b,\cdot]}|y_{\alpha''[X_b,\cdot]})|s_1\right] =$$

$$E\left[(Y_{\alpha' b}/Y_{\alpha'[X_b,\cdot]})(Y_{\alpha'' b}/Y_{\alpha''[X_b,\cdot]})|s_1\right] =$$

$$(Y_{\alpha' b}/Y_{\alpha'[X_b,\cdot]})(Y_{\alpha'' b}/Y_{\alpha''[X_b,\cdot]}) =$$

$$E(y_{\alpha' b}/y_{\alpha'[X_b,\cdot]}|s_1)E(y_{\alpha'' b}/y_{\alpha''[X_b,\cdot]}|s_1) . \quad \therefore$$

$$\text{cov}(y_{\alpha' b}/y_{\alpha'[X_b,\cdot]}, y_{\alpha'' b}/y_{\alpha''[X_b,\cdot]}|s_1) = 0 . \quad \therefore$$

$$\sigma^2(\xi|s_1) = \sum_{\substack{\alpha \\ (Y_{\alpha[X_b,\cdot]}>0)}} \left(\frac{c_{\cdot\cdot}}{n} Y_{\alpha[X_b,\cdot]}\right)^2 \sigma^2(y_{\alpha b}/y_{\alpha[X_b,\cdot]}|s_1) .$$

$$\sigma^2(y_{\alpha b}/y_{\alpha[X_b,\cdot]}|s_1)$$

$$= \sigma^2\left\{E(y_{\alpha b}/y_{\alpha[X_b,\cdot]}|y_{\alpha[X_b,\cdot]})|s_1\right\} + E\left\{\sigma^2(y_{\alpha b}/y_{\alpha[X_b,\cdot]}|y_{\alpha[X_b,\cdot]})|s_1\right\} .$$

$$\sigma^2\left\{E(y_{\alpha b}/y_{\alpha[X_b,\cdot]}|y_{\alpha[X_b,\cdot]})|s_1\right\} = \sigma^2\left\{Y_{\alpha b}/Y_{\alpha[X_b,\cdot]}|s_1\right\} = 0 .$$

$$E\left\{\sigma^2(y_{\alpha b}/y_{\alpha[X_b,\cdot]}|y_{\alpha[X_b,\cdot]})|s_1\right\} =$$

$$E\left\{\frac{Y_{\alpha b}(Y_{\alpha[X_b,\cdot]} - y_{\alpha[X_b,\cdot]})(Y_{\alpha[X_b,\cdot]} - Y_{\alpha b})}{Y^2_{\alpha[X_b,\cdot]}\, y_{\alpha[X_b,\cdot]}(Y_{\alpha[X_b,\cdot]} - 1)} \;|s_1\right\} .$$

$$\therefore \; \sigma^2(\xi|s) =$$

$$\frac{c(c_{\cdot\cdot} - n)(c_{\cdot\cdot} - c)}{n(c_{\cdot\cdot} - 1)}$$

$$+ \; E\left\{ \sum_{\substack{\alpha \\ (Y_{\alpha[X_b,\cdot]}>0)}} \left(\frac{c_{\cdot\cdot}}{n}\right)^2 \frac{Y_{\alpha b}(Y_{\alpha[X_b,\cdot]} - y_{\alpha[X_b,\cdot]})(Y_{\alpha[X_b,\cdot]} - Y_{\alpha b})}{y_{\alpha[X_b,\cdot]}(Y_{\alpha[X_b,\cdot]} - 1)} \right\} .$$

(b) *Cases II, III*

(1) *Cohort / Size Key*

(H , X [at t], S) of a fish caught (in the zone) from Λ during $(t-\delta t/2, t+\delta t/2)$ is a trivariate. Let $p_{\alpha\beta} \equiv P_r(H=H_\alpha , \Gamma=\Gamma_\beta)$,
$\phi_{\beta\alpha} \equiv \Pr(\Gamma=\Gamma_\beta|H=H_\alpha)$, $\chi_{\beta\alpha} \equiv \Pr(\Gamma=\Gamma_\beta|H=H_\alpha , S=S_\beta)$,
$\psi_{\beta\alpha} \equiv \Pr(\Gamma=\Gamma_\beta|H=H_\alpha , X=X_\beta)$, $\gamma_{\alpha\beta} \equiv \Pr(H=H_\alpha|\Gamma=\Gamma_\beta)$.

Let $\mu_{\alpha\beta} \equiv E(c_{\alpha\beta}|\Lambda)$.

Theorem: –

(1) For all size - classes and cohort groups, that can be represented in s (given Λ), $\mu_{\cdot *} = \phi_{**}\mu_{*\cdot} \quad \gamma^{-1}_{**}\mu_{*\cdot}$.

(2) For all size -classes and cohort groups of sex S, that can be represented in s (given Λ), $\mu_{\cdot[*,S]} = \chi_{[*,S]*}\, \mu_{*[\cdot,S]}$, $\gamma^{-1}_{*[*,S]}\, \mu_{*[\cdot,S]}$.

(3) For all size - classes and the two cohort groups of age X at t (i.e. one male cohort and one female cohort), that can be represented in s (given Λ), $\mu_{\cdot[X,*]} = \psi_{[X,*]*}\, \mu_{*[X,\cdot]}$, $\gamma^{-1}_{*[X,*]}\, \mu_{*[X,\cdot]}$.

Proof: —

(1) $p_{\cdot\beta} = \sum_{\alpha} p_{\alpha\cdot}\phi_{\beta\alpha}$, $p_{\alpha\cdot} = \sum_{\beta} p_{\cdot\beta}\gamma_{\alpha\beta}$.

$\therefore p_{\cdot *} = \phi_{**}\, p_{*\cdot}$, $\gamma^{-1}_{**}\, p_{*\cdot}$.

$\therefore E(c_{\cdot *}|c_{\cdot\cdot}) = c_{\cdot\cdot}\, p_{\cdot *} = c_{\cdot\cdot}\, \phi_{**}\, p_{*\cdot} = \phi_{**}\, E(c_{*\cdot}|c_{\cdot\cdot})$.

$\therefore \mu_{\cdot *} = \phi_{**}\, \mu_{*\cdot}$.

Similarly $\mu_{\cdot *} = \gamma^{-1}_{**}\, \mu_{*\cdot}$.

(2) $p_{\cdot[X_\beta,S]} = \sum_{\alpha} p_{\alpha[\cdot,S]}\ \chi_{[X_\beta,S]\alpha}$, $p_{\alpha[\cdot,S]}$

$$= \sum_{X_{\beta(*)}} p_{\cdot[X_{\beta(*)},S]}\ \gamma_{\alpha[X_{\beta(*)},S]} .$$

$\therefore p_{\cdot[*,S]} = \chi_{[*,S]*}\, p_{*[\cdot,S]}$, $\gamma^{-1}_{*[*,S]}\, p_{*[\cdot,S]}$.

By reasoning similar to (1), the result follows.

(3) $p_{\cdot[X,S_\beta]} = \sum_{\alpha} p_{\alpha[X,\cdot]}\psi_{[X,S_\beta]\alpha}$, $p_{\alpha[X,\cdot]} = \sum_{S_\beta} p_{\cdot[X,S_\beta]}\ \gamma_{\alpha[X,S_\beta]}$.

$\therefore p_{\cdot[X,*]} = \psi_{[X,*]*}\, p_{*[X,\cdot]}$, $\gamma^{-1}_{*[X,*]}\, p_{*[X,\cdot]}$.

By reasoning similar to (1), the result follows.

Q.E.D.

ϕ_{**} , $\chi_{[*,S]*}$, $\psi_{[X,*]*}$ and γ^{-1}_{**} , $\gamma^{-1}_{*[*,S]}$, $\gamma^{-1}_{*[X,*]}$ are respectively the direct and the inverse cohort / size keys for S . They depend upon the sampling interval, (the zone) and Λ .

The requirement that the key includes only size - classes and cohort groups that can be represented in S ensures that the matrix has no rows or columns with all zeros.

The direct matrix need not be square.

The inverse matrix must be square to exist. If there are more size classes than cohort groups, the theorem still holds for each square inverse submatrix of order equal to the number of cohort groups.

Expressions for $\phi_{\beta\alpha}$, $\chi_{\beta\alpha}$, $\psi_{\beta\alpha}$, $\gamma_{\alpha\beta}$. — Cohort $\beta(j)$ is recruited at $t-(X_{\beta(j)}-X_r)$. Let $t_{\beta(j)} \equiv \max\,(t-\delta t/2\, ,\, t-(X_{\beta(j)}-X_r))$. Let $\beta_i(j)$ represent the i^{th} fish $\varepsilon\ \beta(j)$. If Γ_β contains only one cohort, then $t_{\beta(1)}$, $\beta_i(1)$ can be written as t_β , β_i respectively.

Let $p_{\beta_i(j)}(\alpha) \equiv \Pr(\beta_i(j)$ caught in H_α during $(t_{\beta(j)}, t+\delta t/2)|\varepsilon \Lambda)$.

Hence $p_{\beta_i(j)}(\alpha) = E(\int q_{\beta_i(j)} dg|\Lambda)$, the integration being for that portion of $(t_{\beta(j)}, t+\delta t/2)$ during which $\beta_i(j)$ is in H_α . (If growth is stochastic given Λ , the portion varies hence the expected value must be taken.)

Let $a_{\beta_i(j)}$ be the variate a_i for $\beta_i(j)$ but at $t_{\beta(j)}$.

Then

$$\phi_{\beta\alpha} = \sum_i a_{\beta_i} p_{\beta_i}(\alpha) \Big/ \sum_{\beta'} \sum_i a_{\beta'_i} p_{\beta'_i}(\alpha) ,$$

$$\chi_{\beta\alpha} = \sum_i a_{\beta_i} p_{\beta_i}(\alpha) \Big/ \sum_{X_{\beta'}} \sum_i a_{[X_{\beta'}, S_\beta]_i} p_{[X_{\beta'}, S_\beta]_i}(\alpha) ,$$

$$\psi_{\beta\alpha} = \sum_i a_{\beta_i} p_{\beta_i}(\alpha) \Big/ \sum_{S_{\beta'}} \sum_i a_{[X_\beta, S_{\beta'}]_i} p_{[X_\beta, S_{\beta'}]_i}(\alpha) ,$$

$$\gamma_{\alpha\beta} = \sum_j \sum_i a_{\beta_i(j)} p_{\beta_i(j)}(\alpha) \Big/ \sum_j \sum_i \sum_{\alpha'} a_{\beta_i(j)} p_{\beta_i(j)}(\alpha') .$$

Sufficient Condition for Key Deterministic (Given R): – ϕ_{**} is to be used in equations of the type $\mu_{\cdot\beta} = \sum_\alpha \phi_{\beta\alpha}\mu_{\alpha\cdot}$ (see above theorem). However, the value of ϕ_{**} used will be (an estimate of) $E(\phi_{**}|$. Similarly for the other keys.

Hence

(1) ϕ_{**} , γ_{**}^{-1} respectively can be regarded as deterministic for a given β if, for all α ,

$\sigma[\phi_{\beta\alpha}|$, $\sigma[(\gamma_{**}^{-1})_{\beta\alpha}| \ll \mu_{\cdot\beta}/\mu_{\cdot\cdot}$.

(2) $\chi_{[*,S]*}$, $\gamma_{*[*,S]}^{-1}$ respectively can be regarded as deterministic for a given β if, for all α ,

$\sigma[\chi_{[X_\beta,S]\alpha}|$, $\sigma[(\gamma_{*[*,S]}^{-1})_{\beta\alpha}| \ll \mu_{\cdot[X_\beta,S]}/\mu_{\cdot[\cdot,S]}$.

(3) $\psi_{[X,*]*}$, $\gamma_{*[X,*]}^{-1}$ respectively can be regarded as deterministic for a given β if, for all α ,

$\sigma[\psi_{[X,S_\beta]\alpha}|$, $\sigma[(\gamma_{*[X,*]}^{-1})_{\beta\alpha}| \ll \mu_{\cdot[X,S_\beta]}/\mu_{\cdot[X,\cdot]}$.

The key is assumed to be deterministic for every cohort,sampling interval for which ξ is to be calculated.

To be able to test this assumption for Case II, where the key is estimated by sampling the catch for cohort / size structure, rather than by calculation (Case III), expressions for the upper bounds of the above standard deviations of the matrix elements are now developed. Growth is assumed deterministic, so the key is regarded as a variate (given R) only through the $a_{\beta_i(j)}$ being variates.

(For Case III, the standard deviations of the key matrix elements will be calculated besides the expected values (given R) (see this Section — Determination of Key). These calculations include an additional source of variation in the key by making the $h_{\beta_i(j)}(X)$ variates (stochastic growth). Then the $p_{\beta_i(j)}(\alpha)$ are variates; and so are the $q_{\beta_i(j)}$ and the $M_{\beta_i(j)}$, thus affecting the probability distributions of the $a_{\beta_i(j)}$ and also causing further variations in the $p_{\beta_i(j)}(\alpha)$.)

Let $N_{\alpha\beta}$ be the number of fish $\varepsilon\ \Gamma_\beta \cap \Lambda$ with non-zero $p_{\beta_i(j)}(\alpha)$. Let $A = \phi_{\beta\alpha}, \chi_{\beta\alpha}$ or $\psi_{\beta\alpha}$

For α , β such that $p_{\beta_i}(\alpha) = 0$ for all i : $E(N_{\alpha\beta}| = 0$. $E(A| = 0$. $\sigma(A| = 0$.

For α , β such that $p_{\beta_i}(\alpha) \neq 0$ for all i : $E(N_{\alpha\beta}| > 0$. $E(A| > 0$. A is the ratio of two summations, each of which can be regarded as containing only terms with non-zero $p_{\beta_i}(\alpha)$. Then $CV[\text{numerator } (A)|$, $CV[\text{denominator } (A)| \lesssim E^{-\frac{1}{2}}(N_{\alpha\beta}|$ (cf. discussion of $CV[r|$ in proof of theorem of Section II - (2)). Hence, by Appendix II - (4), $CV(A| \lesssim E^{-\frac{1}{2}}(N_{\alpha\beta}|$, i.e. $\sigma(A| \lesssim E(A| / E^{\frac{1}{2}}(N_{\alpha\beta}|$.

Hence

(1) ϕ_{**} can be regarded as deterministic for a given β if, for all α such that $E(N_{\alpha\beta}| > 0$ (which is equivalent to $E[\phi_{\beta\alpha}| > 0$), $E(\phi_{\beta\alpha}| / E^{\frac{1}{2}}(N_{\alpha\beta}| << \mu_{\cdot\beta} / \mu_{\cdot\cdot}$.

(2) $\chi_{[*,S]*}$ can be regarded as deterministic for a given β if, for all α such that $E(N_{\alpha[X_\beta,S]}| > 0$ (which is equivalent to $E[\chi_{[X_\beta,S]\alpha}| > 0$), $E(\chi_{[X_\beta,S]\alpha}| / E^{\frac{1}{2}}(N_{\alpha[X_\beta,S]}| << \mu_{\cdot[X_\beta,S]} / \mu_{\cdot[\cdot,S]}$.

(3) $\psi_{[X,*]*}$ can be regarded as deterministic for a given β if, for all α such that $E(N_{\alpha[X,S_\beta]}| > 0$ (which is equivalent to $E[\psi_{[X,S_\beta]\alpha}| > 0$) , $E(\psi_{[X,S_\beta]\alpha}| / E^{\frac{1}{2}}(N_{\alpha[X,S_\beta]}| << \mu_{\cdot[X,S_\beta]} / \mu_{\cdot[X,\cdot]}$.

Likewise, $\sigma(\gamma_{\ell m}| \lesssim \theta_{\ell m}$, where
$\theta_{\ell m} = 0$ if $E(N_{\ell m}| = 0$ (which is equivalent to $E[\gamma_{\ell m}| = 0)$,
$\theta_{\ell m} = E(\gamma_{\ell m}| / E^{\frac{1}{2}}(N_{\ell m}|$ if $E(N_{\ell m}| > 0$.

Hence, by Appendix III,

(1) γ_{**}^{-1} can be regarded as deterministic for a given β if, for all α,

$$\sum_{\ell,m=1}^{n} \left| \frac{\varepsilon_{\ell\alpha}\,\varepsilon_{m\beta}\,\|\,\|\,E(\gamma_{**}|\,\|_{\alpha\beta}\,\|_{\ell m}}{\|\,E(\gamma_{**}|\,\|} - \frac{\|\,E(\gamma_{**}|\,\|_{\alpha\beta}\,\|\,E(\gamma_{**}|\,\|_{\ell m}}{\|\,E(\gamma_{**}|\,\|^2} \right| \theta_{\ell m} \ll \mu_{\cdot\beta} / \mu_{\cdot\cdot} \quad ,$$

where γ_{**} is of order $n \times n$ and $\varepsilon_{ij} = 0 , 1$ if $i = j$, $i \neq j$.

(2) $\gamma_{*[*,S]}^{-1}$ can be regarded as deterministic for a given β if, for all α ,

$$\sum_{\ell,m=1}^{n} \left| \frac{\varepsilon_{\ell\alpha}\,\varepsilon_{m\beta}\,\|\,\|\,E(\gamma_{*[*,S]}|\,\|_{\alpha\beta}\,\|_{\ell m}}{\|\,E(\gamma_{*[*,S]}|\,\|} - \frac{\|\,E(\gamma_{*[*,S]}|\,\|_{\alpha\beta}\,\|\,E(\gamma_{*[*,S]}|\,\|_{\ell m}}{\|\,E(\gamma_{*[*,S]}|\,\|^2} \right| \theta_{\ell[X_{m(*)},S]}$$

$$\ll \mu_{\cdot[X_\beta,S]} / \mu_{\cdot[\cdot,S]} \quad .$$

(3) $\gamma_{*[X,*]}^{-1}$ can be regarded as deterministic for a given β if, for both α ,

$$\sum_{\ell,m=1}^{2} \left| \frac{\varepsilon_{\ell\alpha}\,\varepsilon_{m\beta}\,\|\,\|\,E(\gamma_{*[X,*]}|\,\|_{\alpha\beta}\,\|_{\ell m}}{\|\,E(\gamma_{*[X,*]}|\,\|} - \frac{\|\,E(\gamma_{*[X,*]}|\,\|_{\alpha\beta}\,\|\,E(\gamma_{*[X,*]}|\,\|_{\ell m}}{\|\,E(\gamma_{*[X,*]}|\,\|^2} \right| \theta_{\ell[X,S_m]}$$

$$\ll \mu_{\cdot[X,S_\beta]} / \mu_{\cdot[X,\cdot]} \quad .$$

Examples of Sufficient Condition Fulfilled, and Not Fulfilled: –

Suppose δt is sufficiently small such that, for virtually all i and j , $\beta_i(j)$ is in just one H_α during the whole of $(t_{\beta(j)} , t+\delta t/2)$. Suppose there is no recruitment during $(t-\delta t/2 , t+\delta t/2)$, so each $t_{\beta(j)} = t-\delta t/2$. Suppose each $q_{\beta_i(j)}$ is a positive constant (independent of i , j , t) when $h_{\beta_i(j)}(x)$ is larger than a certain constant (size of first capture), and that $q_{\beta_i(j)}$ is virtually zero when $h_{\beta_i(j)}(x)$ is less. Suppose N_{**} gives reasonable estimates of $E(N_{**}|$, $E(\phi_{**}|$, $E(\gamma_{**}|$.

(1) Let

$$N_{**} = \begin{bmatrix} 0 & 0 & 1\times10^4 & 2\times10^3 & 3\times10^2 & 4\times10 & 5 \\ 0 & 1\times10^5 & 2\times10^4 & 3\times10^3 & 4\times10^2 & 5\times10 & 4 \\ 1\times10^6 & 2\times10^5 & 3\times10^4 & 4\times10^3 & 5\times10^2 & 4\times10 & 3 \\ 2\times10^6 & 3\times10^5 & 4\times10^4 & 5\times10^3 & 4\times10^2 & 3\times10 & 2 \\ 3\times10^6 & 4\times10^5 & 5\times10^4 & 4\times10^3 & 3\times10^2 & 2\times10 & 1 \\ 4\times10^6 & 5\times10^5 & 4\times10^4 & 3\times10^3 & 2\times10^2 & 1\times10 & 0 \\ 5\times10^6 & 4\times10^5 & 3\times10^4 & 2\times10^3 & 1\times10^2 & 0 & 0 \end{bmatrix} .$$

Let $A_{\beta\alpha} \equiv \dfrac{E(\phi_{\beta\alpha}|/E^{\frac{1}{2}}(N_{\alpha\beta}|}{\mu_{\cdot\beta}/\mu_{\cdot\cdot}}$. Then

$$A_{**} \approx \begin{bmatrix} 0 & 0 & 6\times10^{-1} & 3 & 1\times10 & 5\times10 & 2\times10^2 \\ 0 & 2\times10^{-2} & 9\times10^{-2} & 3\times10^{-1} & 1 & 5 & 2\times10 \\ 9\times10^{-4} & 3\times10^{-3} & 1\times10^{-2} & 4\times10^{-2} & 1\times10^{-1} & 5\times10^{-1} & 2 \\ 7\times10^{-4} & 2\times10^{-3} & 7\times10^{-3} & 2\times10^{-2} & 7\times10^{-2} & 2\times10^{-1} & 7\times10^{-1} \\ 6\times10^{-4} & 2\times10^{-3} & 5\times10^{-3} & 1\times10^{-2} & 4\times10^{-2} & 1\times10^{-1} & 3\times10^{-1} \\ 5\times10^{-4} & 1\times10^{-3} & 3\times10^{-3} & 9\times10^{-3} & 2\times10^{-2} & 6\times10^{-2} & 0 \\ 5\times10^{-4} & 1\times10^{-3} & 3\times10^{-3} & 6\times10^{-3} & 1\times10^{-2} & 0 & 0 \end{bmatrix} .$$

Since, for the two youngest cohorts ($\beta = 1$, $\beta = 2$) , $A_{\alpha\beta} \le 10^{-1}$ for all α , ϕ_{1*} and ϕ_{2*} can be regarded as deterministic.

(2) In Example (1), define a new cohort by combining the cohort intervals of the 3rd and 4th youngest cohorts, and group the two size classes for largest size. Then

$$N_{**} = \begin{bmatrix} 0 & 1\times10^5 & 3\cdot5\times10^4 & 7\times10^2 & 9\times10 & 9 \\ 1\times10^6 & 2\times10^5 & 3\cdot4\times10^4 & 5\times10^2 & 4\times10 & 3 \\ 2\times10^6 & 3\times10^5 & 4\cdot5\times10^4 & 4\times10^2 & 3\times10 & 2 \\ 3\times10^6 & 4\times10^5 & 5\cdot4\times10^4 & 3\times10^2 & 2\times10 & 1 \\ 4\times10^6 & 5\times10^5 & 4\cdot3\times10^4 & 2\times10^2 & 1\times10 & 0 \\ 5\times10^6 & 4\times10^5 & 3\cdot2\times10^4 & 1\times10^2 & 0 & 0 \end{bmatrix}, \text{ and}$$

$$A_{**} \approx \begin{bmatrix} 0 & 7\times10^{-3} & 1\times10^{-1} & 2 & 6 & 3\times10 \\ 9\times10^{-4} & 3\times10^{-3} & 1\times10^{-2} & 1\times10^{-1} & 5\times10^{-1} & 2 \\ 7\times10^{-4} & 2\times10^{-3} & 6\times10^{-3} & 7\times10^{-2} & 2\times10^{-1} & 7\times10^{-1} \\ 6\times10^{-4} & 2\times10^{-3} & 5\times10^{-3} & 4\times10^{-2} & 1\times10^{-1} & 3\times10^{-1} \\ 5\times10^{-4} & 1\times10^{-3} & 3\times10^{-3} & 2\times10^{-2} & 6\times10^{-2} & 0 \\ 5\times10^{-4} & 1\times10^{-3} & 2\times10^{-3} & 1\times10^{-2} & 0 & 0 \end{bmatrix}. \text{ So now}$$

ϕ_{1*} , ϕ_{2*} and ϕ_{3*} can be regarded as deterministic.

(3) Let N_{**} be as in Example (1). Let

$$A'_{\alpha\beta} \equiv \left\{ \sum_{\ell,m=1}^{n} \left| \frac{\varepsilon_{\ell\alpha}\varepsilon_{m\beta} \| \, \| E(\gamma_{**}| \, \|_{\alpha\beta} \|_{\ell m}}{\| E(\gamma_{**}| \, \|} - \frac{\| E(\gamma_{**}| \, \|_{\alpha\beta} \| E(\gamma_{**}| \, \|_{\ell m}}{\| E(\gamma_{**}| \, \|^2} \right| \theta_{\ell m} \right\} \Big/ \left\{ \mu_{\cdot\beta} / \mu_{\cdot\cdot} \right\} ,$$

where $\theta_{\ell m}$, n , ε_{ij} are as defined in the previous subsection. The numerator of $A'_{\alpha\beta}$ is calculated by SIGIN (see Appendix III). In this case, all the elements of A'_{**} are > 1 , and most are >> 1 .

(4) In Example (1), group the 5 oldest cohorts and the 5 size classes for largest size. Then $N_{**} = \begin{bmatrix} 6\times10^6 & 1\times10^6 & 1\cdot70\times10^5 \\ 4\times10^6 & 5\times10^5 & 4\cdot32\times10^4 \\ 5\times10^6 & 4\times10^5 & 3\cdot21\times10^4 \end{bmatrix}$,

and $A'_{**} \approx \begin{bmatrix} 6\times10^{-2} & 2 & 9 \\ 1\times10^{-1} & 4 & 2\times10 \\ 7\times10^{-2} & 2 & 9 \end{bmatrix}$. Hence $(\Upsilon^{-1}_{**})_{1*}$ can be regarded as deterministic. All elements of A'_{**} have been reduced.

(5) Let $N_{**} = \begin{bmatrix} 0 & 4{\cdot}64\times10^5 & 4{\cdot}31\times10^5 & 3\times10^5 \\ 1\times10^6 & 9{\cdot}28\times10^5 & 6{\cdot}46\times10^5 & 2\times10^5 \\ 2\times10^6 & 1{\cdot}39\times10^6 & 4{\cdot}31\times10^5 & 1\times10^5 \\ 3\times10^6 & 9{\cdot}28\times10^5 & 2{\cdot}15\times10^5 & 0 \end{bmatrix}$. In this matrix there is less variation over the $N_{\cdot\beta}$ than in Examples (3) and (4). Consequently, the $A'_{\alpha\beta}$ are, in general, reduced:

$$A'_{**} \approx \begin{bmatrix} 4\times10^{-2} & 1\times10^{-1} & 2\times10^{-1} & 4\times10^{-1} \\ 1\times10^{-1} & 3\times10^{-1} & 5\times10^{-1} & 8\times10^{-1} \\ 2\times10^{-1} & 3\times10^{-1} & 5\times10^{-1} & 8\times10^{-1} \\ 7\times10^{-2} & 2\times10^{-1} & 2\times10^{-1} & 3\times10^{-1} \end{bmatrix} .$$

(6) In Example (5), increase every $N_{\alpha\beta}$ by $\times 100$. Then every $A'_{\alpha\beta}$ is reduced by $\times 10^{-1}$. So $(\Upsilon^{-1}_{**})_{1*}$, $(\Upsilon^{-1}_{**})_{2*}$, $(\Upsilon^{-1}_{**})_{3*}$, $(\Upsilon^{-1}_{**})_{4*}$ can be regarded as deterministic.

Examples (1) - (4) illustrate that variation in a key might be reduced by lengthening the size-class intervals, and, in an inverse key, also by grouping cohorts for which ξ is not to be calculated.

If the variation in the key is still too great then the cohort intervals themselves might be lengthened.

Determination of Key: – For Cases II and III, the key must be known for each s sampled. However, for any specific fishery, the number of instances for which the key is to be determined might be reduced by making appropriate assumptions on the functional form of the key. For instance, it might be argued that the key varies with t only seasonally; or, for a non-migratory species, the key might be independent of zone - but this will not in general be true, since in general food supply and hence growth depends upon area.

(1) Calculate Key from Growth, Reproduction and Mortality Parameters (Case III): –

In general, the key depends upon the q_i and the M_i . Hence an iteration procedure is necessary where estimates of the mortality rates are used to estimate the key which is used to obtain better estimates of the mortality rates etc., to convergence. During any step of the iteration, the mortality rates estimated by this model are the functions q , M , and from these $q(h , x , t)$, $M(h , x , t)$ must somehow be deduced. For instance, it might be assumed that $q(h , x , t)$, for given x , t , is a positive constant of zero depending on whether h is greater or less than a certain value, and that $M(h , x , t)$ is independent of h .

A direct key will depend upon the q_i , M_i essentially through the average values q, M whereas an inverse key will depend upon the q_i , M_i essentially only through variations in the $h_i(x)$ for given t . So it is anticipated that the direct key will depend upon $q(h , x , t)$, $M(h , x , t)$ more strongly than will the inverse key. In fact, if $M(h , x , t)$ is independent of h , all members of a cohort are then subjected to virtually the same natural mortality, so the inverse key should be independent of $M(h , x , t)$ to a good approximation, provided also that, for every α , β , $\Pr(H = H_\alpha |$ fish caught $\varepsilon\ \beta(j))$ is the same for all j .

The direct key depends upon the relative numbers in the cohorts at recruitment. The inverse key is independent of these relative numbers, provided again that, for every α , β , $\Pr(H = H_\alpha |$ fish caught $\varepsilon\ \beta(j))$ is the same for all j .

Example (1): Deterministic Growth: – When all fish of one sex (in the zone) $\varepsilon\ \Lambda$ have the same growth curve and the H_α are chosen each to correspond to a unique cohort, then $\chi_{[*,S]*}$ and $\gamma^{-1}_{*[*,S]}$ are both equal to the unit matrix, independent of q and M ; but the other keys depend upon q and M .

Further Examples: –

More complicated keys are calculated using the previous formulae (see this Section - Expressions for $\phi_{\beta\alpha}$, $\chi_{\beta\alpha}$, $\psi_{\beta\alpha}$, $\gamma_{\alpha\beta}$) and computer simulation. The computer generates many possible values of $\phi_{\beta\alpha}$, $\chi_{\beta\alpha}$, $\psi_{\beta\alpha}$ or $\gamma_{\alpha\beta}$, and, for the inverse key, inverts each γ_{**} , $\gamma_{*[*,S]}$ or $\gamma_{*[X,*]}$ generated. From these values the expected values (given R) and the standard deviations of the key matrix elements are calculated.

Two examples are given below, involving respectively stochastic von Bertalanffy growth and stochastic stepwise growth (as exhibited by rock lobster). (This stochastic growth acts as a perturbation, causing $\chi_{[*,S]*}$ and $\gamma^{-1}_{*[*,S]}$ to depart from the unit matrix. Note that this departure will be smaller the smaller the time period, within a cohort interval, over which the cohort is born; for then the size frequency distribution of the cohort will be sharper.) It is not claimed here that the assumptions contained in these examples hold for any definite fishery. The picture presented merely represents a plausible starting point for the application of the model to simulate the cohort/size key. In any particular application, each assumption should be investigated and either verified or modified.

Possible techniques, for measuring the non-mortality parameters involved, are not discussed.

Example (2): Stochastic von Bertalanffy Growth: –

Suppose each individual grows, from birth, along its own von Bertalanffy growth curve, $h = h_\infty(1-e^{-k(x-x_o)})$, till it dies. Suppose that the probability of a fish $\varepsilon\ \beta(j)$ being born with h_∞ , k , x_o in the range $(h_\infty , h_\infty + dh_\infty)$, $(k , k + dk)$, $(x_o , x_o + dx_o)$ is $f(h_\infty , k , x_o)\,dh_\infty\,dk\,dx_o$. (For instance, h_∞ , k and x_o might be independently normally distributed, whence we could write

$$f(h_\infty,k,x_o) = \frac{1}{(2\pi)^{\frac{3}{2}}\sigma(h_\infty)\sigma(k)\sigma(x_o)}\, e^{-\frac{1}{2}\left[\left(\frac{h_\infty - E(h_\infty)}{\sigma(h_\infty)}\right)^2 + \left(\frac{k-E(k)}{\sigma(k)}\right)^2 + \left(\frac{x_o - E(x_o)}{\sigma(x_o)}\right)^2\right]} .$$

In this case there would be six growth parameters for $\beta(j)$, viz $E(h_\infty)$, $E(k)$, $E(x_o)$, $\sigma(h_\infty)$, $\sigma(k)$, $\sigma(x_o)$.)

Let $A(t)dt$ be the probability that a fish $\varepsilon\ \beta(j)$ was born during $(t, t+dt)$ i.e., $\int_{\text{cohort interval}} A(t)dt = 1$. $\beta(j)$ is simulated by giving a fish a birthdate according to the probability law $A(t)$ and growth parameters according to the probability law $f(h_\infty , k , x_o)$. However, a correction must be made for different fish having different chances of survival to recruitment at cohort age X_r . For instance, if $M(h , x , t)$ (for the relevant sex) is a function only of x , written as $M(x)$, these differences in chance of survival will be due only to differences in age at recruitment. In this case the chosen individual is given the probability $e^{-\int_{x_r'}^{x_r} M(x)\,dx}$ of being alive at recruitment, where x_r , x_r' are the respective ages at recruitment of the chosen individual and the youngest possible cohort member. Selection is continued till the required number N_o of members in the cohort (i.e., all alive at recruitment) have been chosen. (N_o for a cohort can be estimated as follows: $\xi/(\hat{q}\ \delta g)$ estimates N for a given t . Using the earliest of these estimates available for the cohort, N_o is estimated by

$$\frac{\xi}{\hat{q}\ \delta g}\ e^{\int_{t_o}^{t} \hat{q}\ dg\ +\ \hat{M}\ dt}$$
.)

The cohort $\beta(j)$ is now allowed to grow from recruitment, the i^{th} individual, say, having probability dP_i of dying during dt .

In this manner, as time progresses, the computer generates and recruits each cohort, and allows each cohort to evolve. Hence Λ is generated. The key is now calculated.

The above process is repeated many times, each time a new possible recruitment set R being generated and allowed to evolve, giving a new possible value of the key.

If R was known, then of course it would not be generated before being allowed to evolve. This shows that the above built in "variations" in R are due to our lack of precise knowledge about R . Because of these variations, the standard deviations of the key matrix elements calculated in this way will overestimate the standard deviations given R . However, provided these estimates are small enough to satisfy the sufficient condition for the key being deterministic given R , not only can the key be regarded as deterministic given R , but also the above uncertainties in R can be neglected when calculating the mean square errors of the estimates of the key matrix elements, namely the $\tau^2(\hat{B}_i)$, in Section VIII.

Example (3): Stochastic Stepwise Growth: —

This example is based on rock lobsters, which retain a constant length except at moulting, when the length increases by a finite amount.

Throughout this paper, for rock lobsters, let the "birthdate" of a lobster be the time when it settles from the plankton to spend the remainder of its life in the benthos. Thus, for instance, the phrase "a lobster is born on a cohort interval" is synonymous with "a lobster settles on a cohort interval" , and age x is measured from settlement.

Assume that the probability that a lobster ε $\beta(j)$ moults, when its age range is $(x , x + dx)$, is of the form $p(x , x_m)dx$, where x_m is the age of the last moult. Let $f(x , \Delta h)d\Delta h$ be the probability that the increment in h for this moult is in the range $(\Delta h , \Delta h + d\Delta h)$.

Let $A(t)dt$ be the probability that a lobster $\varepsilon\ \beta(j)$ was born (settled) during $(t, t+dt)$. $\beta(j)$ is simulated by giving a lobster a birthdate according to the probability law $A(t)$ and allowing it to grow stochastically as in the previous paragraph. As for Example (2), a correction should be made for different fish having different chances of survival to recruitment. This is done by giving the chosen individual a certain probability, e.g., $e^{-\int_{x'_r}^{x_r} M(x)\,dx}$, of entering the cohort (being alive at recruitment). Selection is continued till the required number N_o of members in the cohort have been chosen.

The remainder of the simulation of the key is as for Example (2).

(2) Sample Catch for Cohort/Size Structure to Estimate Key (Case II). — Sampling is not necessarily at the time when the key is to be used, i.e. the catch sampled is not necessarily s.

(2)(a) Sample Catch for Cohort and Size: —

Example: —

Suppose M decreases because predator density is reduced by fishing the predators. Suppose there has always been plenty of food for the prey so their growth characteristics do not change, leaving q unchanged.

So it is necessary to find the new M for the prey.

Suppose $M(h, x, t)$ is independent of h. Group the older cohorts into $\Gamma_{\beta'}$. Suppose, for all fish $\varepsilon\ \Gamma_{\beta'}$, q_i is independent of h and they have ceased to grow; consequently, for every α, $\Pr(H = H_\alpha |$ fish caught $\varepsilon\ \beta'(j))$ is the same for all j. Hence γ_{**}^{-1} is independent of $M(h, x, t)$. Assume also that $q(h, x, t)$ varies with t seasonally. Then γ_{**}^{-1} varies with t only seasonally.

Suppose one had sampled seasonally for cohort and size. As the seasonal function $q(h, x, t)$ and the growth characteristics are unchanged, and as γ_{**}^{-1} is independent of $M(h, x, t)$, then the seasonal function γ_{**}^{-1} is unchanged. So it is not necessary to sample again for cohort to get ξ.

This is so even if cohort strength at recruitment is variable.

Suppose, in the cohort/size sample of a certain season, $y_{\alpha\beta}$ was the frequency of $H = H_\alpha$, $\Gamma = \Gamma_\beta$. Let $\hat{\gamma}_{\alpha\beta} \equiv y_{\alpha\beta}/y_{\cdot\beta}$. Then $E(\hat{\gamma}_{\alpha\beta}) = \gamma_{\alpha\beta}$.

Let $\tau[(\hat{\gamma}_{**}^{-1})_{\beta\alpha}] \equiv E^{\frac{1}{2}}[(\hat{\gamma}_{**}^{-1})_{\beta\alpha} - (\gamma_{**}^{-1})_{\beta\alpha}]^2$. Then by Appendix III,

$$\tau[(\hat{\gamma}_{**}^{-1})_{\beta\alpha}] \leq \sum_{\ell,m=1}^{n} \left| \frac{\varepsilon_{\ell\alpha}\varepsilon_{m\beta} \| \|\gamma_{**}\|_{\alpha\beta} \|_{\ell m}}{\|\gamma_{**}\|} - \frac{\|\gamma_{**}\|_{\alpha\beta} \|\gamma_{**}\|_{\ell m}}{\|\gamma_{**}\|^2} \right| \sigma(\hat{\gamma}_{\ell m}) ,$$

where γ_{**} is of order $n \times n$ and $\varepsilon_{ij} = 0,1$ if $i = j$, $i \neq j$. In this relation, γ_{**} is estimated by $\hat{\gamma}_{**}$. Also,

$\sigma^2(\hat{\gamma}_{\ell m}) = E\left[\sigma^2(\hat{\gamma}_{\ell m}) | y_{\cdot m}\right] = E\left[\frac{\gamma_{\ell m}(1-\gamma_{\ell m})}{y_{\cdot m}}\right]$, which is estimated

by $y_{\ell m}(y_{\cdot m} - y_{\ell m})/y_{\cdot m}^3$.

(2)(b) Sample Catch for Size (and Possibly Sex or Age) and Estimate the Component of Each Cohort in the Size Frequency Function of a Fish Caught: –

Example: –

Assume (for instance) that the size frequency function of a fish caught is $f(h; \omega_*, u_*, \sigma_*) \equiv \sum_{\Gamma_\beta} \frac{\omega_\beta}{\sqrt{2\pi}\,\sigma_\beta} e^{-\frac{1}{2}\left(\frac{h-u_\beta}{\sigma_\beta}\right)^2}$, where $\sum_{\Gamma_\beta} \omega_\beta = 1$.

In a random sample of n fish from the catch let there be y_α with $H = H_\alpha$. Using Langrange's method of undetermined multipliers, ω_* , u_* , σ_* can be estimated by minimising $\sum_\alpha (n\int_{H_\alpha} f(h; \hat{\omega}_*, \hat{u}_*, \hat{\sigma}_*)dh - y_\alpha)^2$ with respect to $\hat{\omega}_*$, $\hat{u}_*$, $\hat{\sigma}_*$, subject to the constraint $\sum_{\Gamma_\beta} \hat{\omega}_\beta = 1$.

$$\text{Let } \phi_{\beta\alpha}(\omega_*, u_*, \sigma_*) \equiv \int_{H_\alpha} \frac{\omega_\beta}{\sqrt{2\pi}\,\sigma_\beta} e^{-\frac{1}{2}\left(\frac{h-u_\beta}{\sigma_\beta}\right)^2} dh \Big/ \int_{H_\alpha} f(h;\omega_*,u_*,\sigma_*)dh .$$

$\phi_{\beta\alpha}$ is estimated by $\phi_{\beta\alpha}(\hat{\omega}_*, \hat{u}_*, \hat{\sigma}_*)$.

The mean square error of $\phi_{\beta\alpha}(\hat{\omega}_*, \hat{u}_*, \hat{\sigma}_*)$ can be found as follows:

y_* has the multinomial distribution

$$z(y_*; \omega_*, u_*, \sigma_*) \equiv \left(\frac{n!}{\prod_\alpha y_\alpha!}\right) \prod_\alpha \left(\int_{H_\alpha} f(h;\omega_*,u_*,\sigma_*)\,dh\right)^{y_\alpha} .$$

$$\tau^2(\phi_{\beta\alpha}(\hat{\omega}_*, \hat{u}_*, \hat{\sigma}_*)) \equiv E(\phi_{\beta\alpha}(\hat{\omega}_*, \hat{u}_*, \hat{\sigma}_*) - \phi_{\beta\alpha}(\omega_*, u_*, \sigma_*))^2$$
$$= \sum_{y'_*} z(y'_*;\omega_*,u_*,\sigma_*)(\phi_{\beta\alpha}(\hat{\omega}'_*,\hat{u}'_*,\hat{\sigma}'_*)-\phi_{\beta\alpha}(\omega_*,u_*,\sigma_*))^2,$$ which

is estimated by $\sum_{y'_*} z(y'_*;\hat{\omega}_*,\hat{u}_*,\hat{\sigma}_*)(\phi_{\beta\alpha}(\hat{\omega}'_*,\hat{u}'_*,\hat{\sigma}'_*)-\phi_{\beta\alpha}(\hat{\omega}_*,\hat{u}_*,\hat{\sigma}_*))^2$. This latter summation is evaluated by the method of Appendix VI.

(ii) *Examples of* ξ *for Cases II, III*

For Cases II, III, when sampling s at least one of age, sex is not sampled. Hence size must be sampled, and the cohort/size key used. ξ estimates μ .

If size alone is sampled, an estimate $\hat{c}_{*.}$, of $c_{*.}$, is obtained. Then $\xi = \phi_{b*}\hat{c}_{*.}$ or $(\gamma^{-1}_{**})_{b*}\hat{c}_{*.}$.

If size and sex are sampled, an estimate $c_{*[\cdot,S_b]}$, of $c_{*[\cdot,S_b]}$, is obtained. Then $\xi = \chi_{b*}\hat{c}_{*[\cdot,S_b]}$ or $(\gamma^{-1}_{*[*,S_b]})_{b*}\hat{c}_{*[\cdot,S_b]}$. (The fish sampled for sex might be a subset of those sampled for size - cf. Section V(a)(ii).)

If size and age are sampled, an estimate $\hat{c}_{*[X_b,\cdot]}$, of $c_{*[X_b,\cdot]}$, is obtained. Then $\xi = \psi_{b*}\hat{c}_{*[X_b,\cdot]}$ or $(\gamma^{-1}_{*[X_b,*]})_{b*}\hat{c}_{*[X_b,\cdot]}$. (The fish sampled for age might be a subset of those sampled for size - cf. Section V(a)(ii).)

(When an inverse method is used and there are more size classes than cohort groups, ξ is the arithmetic mean of variates of one of the above types, each involving a square inverse submatrix of order equal to the number of cohort groups.)

Example (1): — A random sample of size n is taken from s . Let y_α be the frequency of $H = H_\alpha$ in the sample, $\alpha = 1,2$ only. Let $\hat{c}_{\alpha\cdot} \equiv c_{\cdot\cdot}\, y_\alpha / n$. Let $\xi = \phi_{b1}\hat{c}_{1\cdot} + \phi_{b2}\hat{c}_{2\cdot}$. Then

$$E(\xi|s) = \phi_{b1}c_{1\cdot} + \phi_{b2}c_{2\cdot} \quad , \quad \sigma^2(\xi|s) = \frac{(\phi_{b1} - \phi_{b2})^2\, c_{1\cdot}\, c_{2\cdot}\, (c_{\cdot\cdot} - n)}{n(c_{\cdot\cdot} - 1)} .$$

Example (2): — In Section VI is developed a more sophisticated example of ξ for Cases II, III which is applicable, for instance, to the Australian southern rock lobster fishery. In this example, the catch s flows from the zone to markets in clusters which can be conveniently sampled for size and sex. $E(\xi|s)$ and $\sigma^2(\xi|s)$ are functions of $c_{*[\cdot,S_b]}(*)$, $c_{\cdot[\cdot,\bar{S}_b]}(*)$ and the parameters of the sampling procedure, where $C_{\alpha\beta}(J)$ is the number with $H = H_\alpha$, $\Gamma = \Gamma_\beta$ in cluster J , and $\bar{S}_b$ is the opposite sex to S_b .

(c) *General Properties of* ξ

Summarising Cases I, II, III we can write $\xi = \sum B_i \zeta_i$, ζ_i being an estimator of c_i , where

for Case I:
$B_1 = 1$, $c_1 = c$, $i = 1$ only; and

for Cases II, III:
$B_i = \phi_{bi}$ or $(\gamma_{**}^{-1})_{bi}$ and $c_i = c_{i.}$; or

$B_i = \chi_{bi}$ or $(\gamma_{*[*,S_b]}^{-1})_{bi}$ and $c_i = c_{i[\cdot,S_b]}$; or

$B_i = \psi_{bi}$ or $(\gamma_{*[X_b,*]}^{-1})_{bi}$ and $c_i = c_{i[X_b,\cdot]}$.

s is in general partitioned into clusters which are sampled for age, sex and size. $\kappa \equiv E(\xi|s)$ is a function of $c_{**}(*)$ and the parameters of the sampling procedure. However, it is assumed that ζ_i is unbiased given s , so $\kappa = \sum B_i c_i$.

Let $\mu_i \equiv E(c_i|\Lambda)$. $\therefore$
$E(\xi|\Lambda) = E(\kappa|\Lambda) = \sum B_i \mu_i = \mu_{.b} = \mu$ (see theorem of Section V(b)(i)) .

$CV(\mu| = CV(\sum \bar{a}_i q_i| \lesssim E^{-\frac{1}{2}}(N|$ (cf. theorem of Section II - proof of (2)) . Hence, assuming $E^{\frac{1}{2}}(N| \gg 1$, μ can be regarded as deterministic, so we can write $E(\xi| = E(\mu| = \mu$.

$\nu \equiv \sigma^2(\xi|s)$ is a function of $c_{**}(*)$ and the parameters of the sampling procedure.

Since each fish which can contribute to c_i has only a small chance of being caught during δt , and since there are many such fish, $c_i|\Lambda \frown$ Poisson (mean μ_i) . ($\frown$ means "is distributed as".) Then, assuming the c_i to be independent,
$\sigma^2(\xi|\Lambda) = \sigma^2(\kappa|\Lambda) + E(\nu|\Lambda) = \sum B_i^2 \mu_i + E(\nu|\Lambda)$.

Since μ and the B_i are regarded as deterministic,
$\sigma^2(\xi| = \sigma^2(\mu| + E[\sigma^2(\xi|\Lambda)| = \sum B_i^2 E(\mu_i| + E(\nu|$.

TABLE 1

24 LANDINGS OF *JASUS NOVAEHOLLANDIAE* HOLTHUIS

$c_{..}(J)$	$c_{i[\cdot,S_b]}(J)$	$c_{..}(J)$	$c_{i[\cdot,S_b]}(J)$	$c_{..}(J)$	$c_{i[\cdot,S_b]}(J)$	$c_{..}(J)$	$c_{i[\cdot,S_b]}(J)$
163	14	117	15	310	34	334	31
51	4	80	5	287	31	279	29
94	4	55	6	275	35	178	21
25	2	82	10	387	42	219	21
89	10	173	16	531	51	408	34
41	4	142	7	472	56	504	60

VI. SCHEME FOR SAMPLING CATCH FOR SIZE

Suppose the catch s flows from the zone to markets in K clusters which can be conveniently sampled for size and sex. For instance, for the Australian southern rock lobster fishery such clusters are "landings", a landing being the catch from one or more boats unloading at one site more or less simultaneously. Thus the techniques of cluster sampling (see, for instance, Cochran 1963) can be applied to find estimators ζ_i such that $E(\zeta_i|s) = c_i = c_{i[\cdot,S_b]}$. It is also desired to find an expression for $\nu \equiv \sigma^2(\xi|s)$, where $\xi \equiv \sum B_i\zeta_i$.

Assume that clusters are, in effect, selected at random from the K clusters of s . Suppose if cluster J is selected then n_J fish are selected at random from cluster J and the size class and the sex of each is determined. Make n_J proportional to $c_{..}(J)$, say $n_J = fc_{..}(J)$. For the j^{th} cluster selected let $c_{i\beta}(j)'$, n_j' be the respective values of $c_{i\beta}(J)$, n_J ; and let y_i be the number with $H = H_i$, $S = S_b$ in the sample of size n_j' .

Suppose k clusters are selected. Let $c'_{i\beta} = \sum_{j=1}^{k} c_{i\beta}(j)'$.

Expected total number measured from s is $\bar{n} = kfc_{..}/K$. Let $f_{\bar{n}} \equiv \bar{n}/c_{..}$ and $f_k \equiv k/K$. $\therefore f = f_{\bar{n}}/f_k$. (Note $f_k \geq f_{\bar{n}}$.) If $f_{\bar{n}}$ and f_k are decided, then f is determined; and, knowing K approximately, k is determined.

Let $\zeta_i = \frac{c_{..}}{c'_{..}} \sum_{j=1}^{k} c_{..}(j)' \frac{y_i}{n_j'}$. ζ_i can be estimated if w_j' , $W(j)'$, W' , W , the weights corresponding to n_j' , $c_{..}(j)'$, $c'_{..}$, $c_{..}$ respectively, are determined and if it can be assumed that $c_{..}(j)'/n_j' \approx W(j)'/w_j'$ and $c_{..}/c'_{..} \approx W/W'$. (Replacing the ratios of numbers by the ratios of weights gives the estimator used by Stark and Halden 1966.)

Let $BS(\zeta_i|s) \equiv E(\zeta_i|s) - c_i$. If $|BS(\zeta_i|s)/\sigma(\zeta_i|s)| << 1$ for all $f_{\bar{n}}$, f_k then ζ_i is virtually unbiased.

For instance, Table 1 shows, for each of 24 landings, the estimated total number of southern rock lobster caught and the number of males in the length class 126-130mm. These landings were selected from the catch taken in the zone area bounded by the Victorian coast, long. 142°E, lat. 40°S, long. 141°E during the sampling interval, August 1969. (From available data, these data were chosen for this illustration because of the relatively large sample sizes. In fact, the taking of females is prohibited during this month.)

TABLE

$BS(\zeta_i|s) / \sigma(\zeta_i|s)$ FOR

	k, f_k					
$f_{\bar{n}}$	2,·08	4,·17	6,·25	8,·33	10,·42	12,·50
·05	-·20					
·10	—	-·03	-·03	-·05	-·10	-·04
·15	—	-·04				
·20	—	—	-·05	-·07	-·15	-·06
·25	—	—	-·05			
·30	—	—	—	-·11	-·20	-·07
·40	—	—	—	—	-·23	-·09
·50	—	—	—	—	—	-·12
·55	—	—	—	—	—	—
·60	—	—	—	—	—	—
·65	—	—	—	—	—	—
·70	—	—	—	—	—	—
·75	—	—	—	—	—	—
·80	—	—	—	—	—	—
·90	—	—	—	—	—	—
·95	—	—	—	—	—	—

Regarding these 24 landings as the whole of s (in fact, s is larger), a computer was used to repeatedly select k landings at random and hence estimate $BS(\zeta_i|s)$ for various f_k using $E(\zeta_i|s) = c_{..}E(c'_{i[\cdot,S_b]}/c'_{..}|s)$, and $\sigma(\zeta_i|s)$ for various $f_{\bar{n}}$, f_k from

$$\sigma^2(\zeta_i|s) = c_{..}^2\,\sigma^2\left\{\frac{c'_{i[\cdot,S_b]}}{c'_{..}}\Big|s\right\} + c_{..}^2\left(\frac{1}{f}-1\right)E\left\{\frac{1}{c'^2_{..}}\sum_{j=1}^{k} c_{..}(j)'\,S_j'^2\,\Big|s\right\} ,$$

where $S_J^2 = \dfrac{c_{i[\cdot,S_b]}(J)}{c_{..}(J)-1}\left(1-\dfrac{c_{i[\cdot,S_b]}(J)}{c_{..}(J)}\right)$ and $S_j'^2$ is the value of S_J^2 for the j^{th} cluster selected. The program is listed in Appendix VII. Table 2 shows the resulting values of $BS(\zeta_i|s)/\sigma(\zeta_i|s)$.

2

LANDINGS OF TABLE 1

k, f_k: 14,·58	16,·67	18,·75	20,·83	22,·92	24,1·00	$f_{\bar{n}}$
						·05
–·04	·03	·00	–·04	·00	·00	·10
						·15
–·06	·04	·00	–·05	·01	·00	·20
						·25
–·07	·06	·00	–·07	·01	·00	·30
–·10	·07	·00	–·09	·01	·00	·40
–·13	·09	·00	–·11	·01	·00	·50
–·13						·55
—	·10	·00	–·14	·02	·00	·60
—	·11					·65
—	—	·01	–·17	·02	·00	·70
—	—	·01				·75
—	—	—	–·24	·03	·00	·80
—	—	—	—	·04	·00	·90
—	—	—	—	—	·00	·95

$\sigma(\zeta_i|s)$ cannot be estimated from a sample. So

let $\zeta_i^* \equiv \frac{K}{k}\sum_{j=1}^{k} c_{..}(j)' \frac{y_j}{n_j'}$. $E(\zeta_i^*|s) = c_i$ (unbiased).

$$\sigma^2(\zeta_i^*|s) = KS^2_{(B)}\left(\frac{1}{f_k}-1\right) + \left\{\sum_{J=1}^{K} c_{..}(J)S_J^2\right\}\left(\frac{1}{f_{\bar{n}}}-\frac{1}{f_k}\right), \text{ where}$$

$S^2_{(B)} = \frac{1}{K-1}\sum_{J=1}^{K}(c_{i[\cdot,S_b]}(J) - c_i/K)^2$. $\sigma^2(\zeta_i^*|s)$ can be estimated

if S_J^2 does not vary much with J and is consequently replaced

by $S^2_{(W)} = \frac{1}{K}\sum_{J=1}^{K} S_J^2$. For instance, the value of $\left\{\frac{1}{K}\sum_{J=1}^{K}(S_J^2 - S^2_{(W)})^2\right\}^{\frac{1}{2}} / S^2_{(W)}$

calculated for the above 24 landings is 0·21. Then

$\sigma^2(\zeta_i^*|s) \approx KS^2_{(B)}\left(\frac{1}{f_k}-1\right) + c_{..}S^2_{(W)}\left(\frac{1}{f_{\bar{n}}}-\frac{1}{f_k}\right)$. This formula was used to find

TABLE

$\sigma(\zeta_i^*|s) \,/\, \sigma(\zeta_i|s)$ FOR

		k, f_k					
		2,·08	4,·17	6,·25	8,·33	10,·42	12,·50
$f_{\bar{n}}$	·05	2·7					
	·10	—	2·7	2·3	1·9	1·8	1·6
	·15	—	3·2				
	·20	—	—	3·5	2·8	2·6	2·2
	·25	—	—	4·0			
	·30	—	—	—	4·2	3·2	2·6
	·40	—	—	—	—	3·6	3·3
	·50	—	—	—	—	—	4·4
	·55	—	—	—	—	—	—
	·60	—	—	—	—	—	—
	·65	—	—	—	—	—	—
	·70	—	—	—	—	—	—
	·75	—	—	—	—	—	—
	·80	—	—	—	—	—	—
	·90	—	—	—	—	—	—
	·95	—	—	—	—	—	—

$\sigma(\zeta_i^*|s)$ for these landings (regarded again as the whole of s) for various $f_{\bar{n}}$, f_k . The resulting values of $\sigma(\zeta_i^*|s)/\sigma(\zeta_i|s)$, shown in Table 3, are seen to be ≥ 1 . So in this example $\sigma(\zeta_i^*|s)$ can be regarded as an upper bound to $\sigma(\zeta_i|s)$. It is anticipated that this will be the case whenever the proportion of a size-sex class in a cluster varies, from cluster to cluster, much less than does the number in the cluster.

From a random sample of k landings from s , $\sigma^2(\zeta_i^*|s)$ can be estimated as a function of $f_{\bar{n}}$, f_k . For let

$$s^2_{i(B)} \equiv \frac{1}{k-1} \sum_{j=1}^{k} (c_{i[\cdot,S_b]}(j)' - c'_{i[\cdot,S_b]}/k)^2 \;,\quad s_j^2 \equiv \frac{y_j}{n_j'-1}\left(1 - \frac{y_j}{n_j'}\right) ,$$

$$s^2_{i(W)} \equiv \frac{1}{k} \sum_{j=1}^{k} s_j^2 \;.\; \text{Then } E(s^2_{i(B)}|s) = S^2_{(B)} \;,\; E(s^2_{i(W)}|s) = S^2_{(W)} \;.$$

3

LANDINGS OF TABLE 1

$k,\ f_k$ → 14,·58	16,·67	18,·75	20,·83	22,·92	24,1·00	$f_{\bar{n}}$
						·05
1·4	1·3	1·2	1·1	1·1	1·0	·10
						·15
1·9	1·7	1·5	1·3	1·1	1·0	·20
						·25
2·2	2·0	1·7	1·5	1·2	1·0	·30
2·8	2·4	2·0	1·7	1·4	1·0	·40
3·6	3·1	2·4	2·0	1·5	1·0	·50
3·7						·55
—	3·3	3·0	2·4	1·7	1·0	·60
—	3·5					·65
—	—	3·6	2·8	2·0	1·0	·70
—	—	4·1				·75
—	—	—	3·8	2·5	1·0	·80
—	—	—	—	3·6	1·0	·90
—	—	—	—	—	1·0	·95

$$\nu^{\frac{1}{2}} \le \sum_i |B_i| \sigma(\zeta_i | s) \quad \text{by Appendix VIII,}$$

$$\le \sum_i |B_i| \sigma(\zeta_i^* | s) \ .$$ So an estimate of the upper bound

of $\nu^{\frac{1}{2}}$ is $\sum_i |B_i| \left\{K s^2_{i(B)} \left(\frac{1}{f_k} - 1\right) + c_{..}\, s^2_{i(W)} \left(\frac{1}{f_{\bar{n}}} - \frac{1}{f_k}\right)\right\}^{\frac{1}{2}}$. This formula can be used in finding confidence ranges of q , M for various amounts of sampling (i.e. for various $f_{\bar{n}}$, f_k) .

VII. $\hat{L}$, THE ESTIMATOR OF L

(i)

$$E[\log\left(\frac{\xi'/\delta g'}{\xi''/\delta g''}\right)| = \log\left\{\frac{E(\xi'/\delta g'|}{E(\xi''/\delta g''|}\right\} - \tfrac{1}{2}CV^2[\xi'| + \tfrac{1}{2}CV^2[\xi''| + O\{CV(\xi'| + CV(\xi''|\}^3$$

$$= L - \tfrac{1}{2}CV^2[\xi'| + \tfrac{1}{2}CV^2[\xi''| + O\{CV(\xi'| + CV(\xi''|\}^3$$

. The leading terms in the bias tend to cancel (hence it is possible to have bias $\sim \{CV[\xi'| + CV[\xi''|\}^3$) . The above equation cannot hold if $\frac{\xi'/\delta g'}{\xi''/\delta g''}$ can be infinite. This situation is avoided by not allowing the variate ξ to take the value zero. This would be a rare event if $E(\xi| \gg 1$ and its omission will have neglible effect on the parameters of the distribution of ξ . So it is assumed that $E(\xi| \gg 1$ (i.e. $\mu \gg 1$) .

$$\sigma^2[\log\left(\frac{\xi'/\delta g'}{\xi''/\delta g''}\right)| = CV^2(\xi'| + CV^2(\xi''| - 2\rho(\xi',\xi''|CV(\xi'|CV(\xi''| + O\{CV(\xi'| + CV(\xi''|\}^3$$

$$= CV^2(\xi'| + CV^2(\xi''| + O\{CV(\xi'| + CV(\xi''|\}^3$$

, for, as s″ is only very weakly correlated with s′ , $|\rho(\xi',\xi''|| \ll 1$.

(ii) $\frac{\sum B_i^2\zeta_i+\nu}{\xi^2}$ *as an Estimator of* $CV^2(\xi|$

$$E(\sum B_i^2\zeta_i+\nu| = \sigma^2(\xi| \quad . \quad E(\xi^2| = E^2[\xi|(1+CV^2[\xi|) \quad . \quad \therefore$$

$$E(\sum B_i^2\zeta_i+\nu|/E(\xi^2| = CV^2(\xi| + O(CV^4[\xi|) \; ..$$

Let $BS\left(\frac{\sum B_i^2\zeta_i+\nu}{\xi^2}\right| \equiv E\left(\frac{\sum B_i^2\zeta_i+\nu}{\xi^2}\right| - \frac{E(\sum B_i^2\zeta_i+\nu|}{E(\xi^2|}$. Then, by Appendix II(1),

$$BS\left(\frac{\sum B_i^2\zeta_i+\nu}{\xi^2}\right| \le CV(\xi^2|\sigma\left(\frac{\sum B_i^2\zeta_i+\nu}{\xi^2}\right| \sim CV(\xi|\sigma\left(\frac{\sum B_i^2\zeta_i+\nu}{\xi^2}\right| \; .$$

By Appendix II(3),

$$\tau\left(\frac{\sum B_i^2\zeta_i+\nu}{\xi^2}\right| = \frac{E(\sum B_i^2\zeta_i+\nu|}{E(\xi^2|}\Big\{CV^2(\sum B_i^2\zeta_i+\nu| + CV^2(\xi^2| - 2\rho(\sum B_i^2\zeta_i+\nu,\xi^2|\,CV(\sum B_i^2\zeta_i+\nu|CV(\xi^2| + O[CV(\xi^2| + CV(\sum B_i^2\zeta_i+\nu|]^3\Big\}^{\frac{1}{2}} \; .$$

$CV(\sum B_i^2\zeta_i+\nu) \le CV(\sum B_i^2\zeta_i)+CV(\nu)$ since $E(\sum B_i^2\zeta_i) > 0$ and $E(\nu) \ge 0$. (When $E(\nu) = 0$ then Γ is determined for each fish ε S so $\nu \equiv 0$, and we can write $CV(\nu) = 0$.) Now $CV(\sum B_i^2\zeta_i) \sim CV(\sum B_i\zeta_i) = CV(\xi)$, since both $CV^2(\sum B_i^2\zeta_i)$ and $CV^2(\sum B_i\zeta_i)$ are of the form $\sum_{i,j}\omega_{ij}\mathrm{cov}(\zeta_i,\zeta_j) / \sum_{i,j}\omega_{ij}E(\zeta_i)E(\zeta_j)$ where $\sum_{i,j}\omega_{ij} = 1$. Also, ν is a function of $c_{**}(*)$. Let $N_{\alpha\beta}(J)$ be the number ε Λ that can contribute to $c_{\alpha\beta}(J)$. If $N_{\alpha\beta}(J) \gg 1$, then $c_{\alpha\beta}(J)|\Lambda \frown$ Poisson (mean $\mu_{\alpha\beta}(J)$, say) (cf. Section V(c)). Hence, if $E[N_{\alpha\beta}(J)] \gg 1$,

$$CV^2[C_{\alpha\beta}(J)] = CV^2[\mu_{\alpha\beta}(J)] + E^{-1}[\mu_{\alpha\beta}(J)]$$

$$\le E^{-1}[N_{\alpha\beta}(J)] + E^{-1}[\mu_{\alpha\beta}(J)] \quad \text{(cf. Section V(c))}$$

$\ll 1$ provided $E[\mu_{\alpha\beta}(J)] \gg 1$. So, if $E[\mu_{\alpha\beta}(J)] \gg 1$ (which implies $E[N_{\alpha\beta}(J)] \gg 1$) , all α , β , J , then it is reasonable to assume that $CV(\nu) \ll 1$, say $CV(\nu) \lesssim CV(\xi)$. So $CV(\sum B_i^2\zeta_i + \nu) \lesssim CV(\xi)$.

$$\therefore \tau\left(\frac{\sum B_i^2\zeta_i+\nu}{\xi^2}\right) \sim CV^3(\xi) .$$

$$\therefore \sigma\left(\frac{\sum B_i^2\zeta_i+\nu}{\xi^2}\right) \sim CV^3(\xi) .$$

$$\therefore E\left(\frac{\sum B_i^2\zeta_i+\nu}{\xi^2}\right) = CV^2(\xi) + O(CV^4[\xi]) .$$

(iii)

Let B_i' , ζ_i' , ν' and B_i'' , ζ_i'' , ν'' be respectively the values of B_i , ζ_i , ν corresponding to s' and s" .

Let $\hat{K}_* \equiv (\zeta_1', \zeta_2', \ldots, \zeta_1'', \zeta_2'', \ldots, \nu', \nu'')$. Let

$$\ell(\hat{K}_*, B_*', B_*'') \equiv \log\left\{\frac{\xi'/\delta g'}{\xi''/\delta g''}\right\} + \frac{1}{2}\frac{\sum(B_i')^2\zeta_i'+\nu'}{(\xi')^2} - \frac{1}{2}\frac{\sum(B_i'')^2\zeta_i''+\nu''}{(\xi'')^2} .$$ Then

$E[\ell(\hat{K}_*, B_*', B_*'')] = L + O\{CV(\xi') + CV(\xi'')\}^3$. However, the estimator of L used will in general be of the form $\hat{L} \equiv \ell(\hat{K}_*, \hat{B}_*', \hat{B}_*'')$ where $\hat{B}_*$, $\hat{B}_*'$, $\hat{B}_*''$ are estimators of B_* , B_*' , B_*'' respectively. (For Case I, $\hat{B}_* \equiv B_* = 1$.)

VIII. CALCULATION OF $\tau^2(\hat{L}) \equiv E(\hat{L}-L)^2$

$$\sigma^2[\ell(\hat{K}_*, B'_*, B''_*)| = CV^2(\xi'| + CV^2(\xi''| + O\{CV(\xi'| + CV(\xi''|\}^3 \quad . \quad \therefore$$

$$\tau^2[\ell(\hat{K}_*, B'_*, B''_*)| \equiv E[\ell(\hat{K}_*, B'_*, B''_*) - L|^2$$

$$= CV^2(\xi'| + CV^2(\xi''| + O\{CV(\xi'| + CV(\xi''|\}^3 \quad . \quad \therefore$$

$\tau^2[\ell(\hat{K}_*, B'_*, B''_*)|$ is estimated by $\dfrac{\sum(\hat{B}'_i)^2\zeta'_i+\nu'}{(\xi')^2} + \dfrac{\sum(\hat{B}''_i)^2\zeta''_i+\nu''}{(\xi'')^2}$. In this expression the terms in ν', ν'' are due to the catch being sampled and the other two terms are due to the catch, for a given effort, being a variate.

(a) Case I

$$E\left[\frac{\sum(B'_i)^2\zeta'_i+\nu'}{(\xi')^2} + \frac{\sum(B''_i)^2\zeta''_i+\nu''}{(\xi'')^2}\right| = \tau^2(\hat{L}| + O\{CV(\xi'| + CV(\xi''|\}^3 \quad .$$

$$CV\left[\frac{\sum(B'_i)^2\zeta'_i+\nu'}{(\xi')^2} + \frac{\sum(B''_i)^2\zeta''_i+\nu''}{(\xi'')^2}\right| \lesssim CV(\xi'| + CV(\xi''| \quad .$$

(b) Cases II, III

$\hat{L}$ is derived from $\hat{K}_*$ and $\hat{B}'_*$, $\hat{B}''_*$ (Fig. 6). Let

$\tau^2(\hat{B}_i) \equiv E(\hat{B}_i - B_i)^2$, $K_i \equiv E(\hat{K}_i|$. Now

$$\hat{L} = \ell(\hat{K}_*, B'_*, B''_*) + \ell(\hat{K}_*, \hat{B}'_*, \hat{B}''_*) - \ell(\hat{K}_*, B'_*, B''_*) \quad . \quad \therefore$$

$\tau(\hat{L}) \le \tau[\ell(\hat{K}_*, B'_*, B''_*)| + E^{\frac{1}{2}}\{z(\hat{K}_*, \hat{B}'_*, \hat{B}''_*)\}^2$, by Appendix VIII,

where $z(\hat{K}_*, \hat{B}'_*, \hat{B}''_*) \equiv \ell(\hat{K}_*, \hat{B}'_*, \hat{B}''_*) - \ell(\hat{K}_*, B'_*, B''_*)$. Note

$z(\hat{K}_*, B'_*, B''_*) \equiv 0$. Hence

$$E\{z(\hat{K}_*, \hat{B}'_*, \hat{B}''_*)\}^2 = \tau^2\{z(\hat{K}_*, \hat{B}'_*, \hat{B}''_*)\} \equiv E\{z(\hat{K}_*, \hat{B}'_*, \hat{B}''_*) - z(K_*, B'_*, B''_*)\}^2 \quad .$$

By Appendix IV,

$$\tau\{z(\hat{K}_*, \hat{B}'_*, \hat{B}''_*)\} \le \sum\left|\frac{\partial\ell(K_*, B'_*, B''_*)}{\partial\hat{B}'_i}\right|\tau(\hat{B}'_i) + \sum\left|\frac{\partial\ell(K_*, B'_*, B''_*)}{\partial\hat{B}''_i}\right|\tau(\hat{B}''_i)$$
$$+ O[\textstyle\sum\tau(\hat{B}'_i) + \sum\tau(\hat{B}''_i) + \sum\sigma(\hat{K}_i|]^2 \quad ;$$

$$\therefore \tau(\hat{L}) \le \tau[\ell(\hat{K}_*, B'_*, B''_*)| + \sum\left|\frac{\partial\ell(K_*, B'_*, B''_*)}{\partial\hat{B}'_i}\right|\tau(\hat{B}'_i) + \sum\left|\frac{\partial\ell(K_*, B'_*, B''_*)}{\partial\hat{B}''_i}\right|\tau(\hat{B}''_i)$$
$$+ O[\textstyle\sum\tau(\hat{B}'_i) + \sum\tau(\hat{B}''_i) + \sum\sigma(\hat{K}_i|]^2 \quad .$$

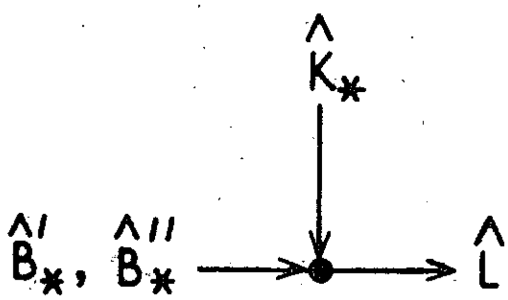

Fig. 6. Flow for derivation of $\hat{L}$ from $\hat{K}_*$ and $\hat{B}'_*$, $\hat{B}''_*$ for Cases II, III.

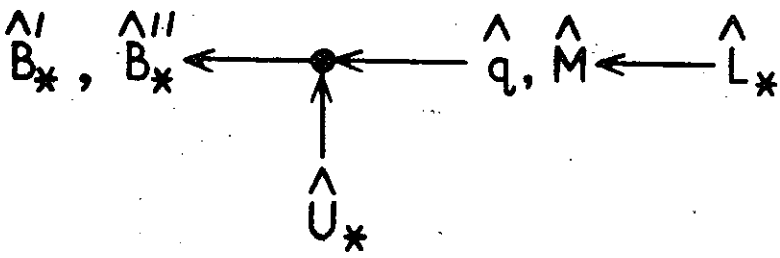

Fig. 7. Flow for derivation of $\hat{B}'_*$, $\hat{B}''_*$ from $\hat{L}_*$ and $\hat{U}_*$ for Case III.

This formula is used to find an upper bound of $\tau(\hat{L})$. The various terms are estimated as follows:

An estimate has already been given for $\tau[\ell(\hat{K}_*,B'_*,B''_*)|$.

$\frac{\partial \ell(K_*,B'_*,B''_*)}{\partial \hat{B}'_i}$ is estimated by

$$\frac{\partial \ell(\hat{K}_*,\hat{B}'_*,\hat{B}''_*)}{\partial \hat{B}'_i} = \frac{\zeta'_i}{\hat{\xi}'} + \frac{\hat{B}'_i\zeta'_i}{(\hat{\xi}')^2} - \frac{(\sum(\hat{B}'_j)^2\zeta'_j+\nu')\zeta'_i}{(\hat{\xi}')^3} \quad \text{, where}$$

$\hat{\xi}' \equiv \sum \hat{B}'_i\zeta'_i$.

$\frac{\partial \ell(K_*,B'_*,B''_*)}{\partial \hat{B}''_i}$ is estimated by

$$\frac{\partial \ell(\hat{K}_*,\hat{B}'_*,\hat{B}''_*)}{\partial \hat{B}''_i} = -\frac{\zeta''_i}{\hat{\xi}''} - \frac{\hat{B}''_i\zeta''_i}{(\hat{\xi}'')^2} + \frac{(\sum(\hat{B}''_j)^2\zeta''_j+\nu'')\zeta''_i}{(\hat{\xi}'')^3} \quad \text{, where}$$

$\hat{\xi}'' \equiv \sum \hat{B}''_i\zeta''_i$.

(i) *Case II*

Examples have previously been given for the evaluation of $\tau(\hat{B}_i)$ for Case II (see Section V(b)(i)).

(ii) *Case III*

For Case III, $\tau(\hat{B}_i)$ is evaluated (with $\tau(\hat{L})$) as follows:

$\hat{B}'_*$, $\hat{B}''_*$ are derived from $\hat{L}_*$, where $\hat{L}_j$ refers to the j^{th} cohort-time pair, and from an estimate $\hat{U}_*$ of the relevant non-mortality parameters U_* (Fig. 7).

Examples of U_* are afforded by Examples (2), (3) of the calculation of the key in Section V(b)(i). In Example (2), the U_j are N_o , for each of the various cohorts that can be represented in s' and s", and the parameters of the functions A(t) and $f(h_\infty, k, x_o)$. For instance, the parameters of the latter are $E(h_\infty)$, $E(k)$, $E(x_o)$, $\sigma(h_\infty)$, $\sigma(k)$, $\sigma(x_o)$. In Example (3), the U_j are the N_o and the parameters of the functions A(t) , $p(x, x_m)$, $f(x, \Delta h)$.

An upper bound to $\tau(\hat{B}'_i)$ is given by

$$\tau(\hat{B}'_i) \lesssim \sum_j \left| \frac{\partial \hat{B}'_i(L_*, U_*)}{\partial \hat{L}_j} \right| \tau(\hat{L}_j) + \sum_j \left| \frac{\partial \hat{B}'_i(L_*, U_*)}{\partial \hat{U}_j} \right| \tau(\hat{U}_j)$$, by Appendix IV.

Similarly for $\tau(\hat{B}''_i)$.

$\frac{\partial \hat{B}'_i}{\partial \hat{L}_j}(L_*, U_*)$ is estimated by $\frac{\partial \hat{B}'_i}{\partial \hat{L}_j}(\hat{L}_*, \hat{U}_*)$ which is evaluated numerically by varying $\hat{L}_j$ whilst keeping the other $\hat{L}_k$, and $\hat{U}_*$, constant. $\frac{\partial \hat{B}'_i}{\partial \hat{U}_j}(L_*, U_*)$ is estimated by $\frac{\partial \hat{B}'_i}{\partial \hat{U}_j}(\hat{L}_*, \hat{U}_*)$ which is evaluated numerically by varying $\hat{U}_j$ whilst keeping the other $\hat{U}_k$, and $\hat{L}_*$, constant.

First approximations to the upper bounds of the $\tau(\hat{L})$ are found by substituting $\tau[\ell_j(\hat{K}_*, B'_*, B''_*)]$ for $\tau(\hat{L}_j)$ in such expressions as above for the upper bound of $\tau(\hat{B}'_i)$, and then substituting the resulting values in the expressions for the upper bounds of the $\tau(\hat{L})$. These first approximations are then substituted in the above such expressions, etc., to obtain the second approximations. This procedure is repeated to convergence.

N_o, when not estimated independently of $\hat{q}, \hat{M}$, but as indicated in Example (2) of the calculation of the key in Section V(b)(i), is derived from $\hat{\xi} \equiv \sum \hat{B}_k \zeta_k$ (for some sampling interval) and $\hat{L}_*$ (to estimate $\hat{q}, \hat{M}$). In this case, by Appendix IV, an estimate of an upper bound of the root mean square error of the estimator $\hat{N}_o$ is given by

$$\tau(\hat{N}_o) \equiv E^{\frac{1}{2}}(\hat{N}_o - N_o)^2 \leq \frac{\hat{N}_o(\hat{\xi}, \hat{L}_*)}{\hat{\xi}} \tau(\hat{\xi}) + \sum_k \left| \frac{\partial \hat{N}_o(\hat{\xi}, \hat{L}_*)}{\partial \hat{L}_k} \right| \tau(\hat{L}_k) ,$$

where $\tau^2(\hat{\xi}) \equiv E(\hat{\xi} - \mu)^2$. Writing $\hat{\xi} = \xi + (\hat{\xi} - \xi)$ and applying Appendices VIII, IV gives
$\tau(\hat{\xi}) \leq \sigma(\xi| + \sum \mu_k \tau(\hat{B}_k)$ which is estimated by
$(\sum \hat{B}_k^2 \zeta_k + \nu)^{\frac{1}{2}} + \sum \zeta_k \tau(\hat{B}_k)$. $\frac{\partial \hat{N}_o(\hat{\xi}, \hat{L}_*)}{\partial \hat{L}_k}$ is estimated numerically by varying $\hat{L}_k$. These expressions for the upper bounds of the $\tau(\hat{N}_o)$ are incorporated into the above iteration for the upper bounds of the $\tau(\hat{L})$; the initial substitution for $\tau(\hat{N}_o)$ omits terms in $\tau(\hat{B}_k)$.

IX. CONDITIONS UNDER WHICH $\tau^2(\hat{L})$ FOR COHORT b AND TIME PAIR t′ , t″ CAN BE CALCULATED BY THIS MODEL

These conditions are that, for cohort b and for each of the two sampling intervals $\delta t'$, $\delta t''$:

(1) $q\,\delta g + M\,\delta t \ll 1$.

(2) $E(\mu| \gg 1$.

(3) (For Cases II, III) the cohort/size key can be regarded as deterministic.

Note that (1) and (2) imply $E^{\frac{1}{2}}(N''| \gg 1$, and that (2) implies $E(N' - N''| \gg 1$.

(1), (2), (3) are initially assumed to hold, the consistency of this assumption being checked on calculating $\hat{q}, \hat{M}$.

If $\hat{\xi} \equiv \sum \hat{B}_i \zeta_i \gg 1$, (2) is assumed to hold.

To test (3), $\mu/\mu_{.}$ is estimated by $\hat{\xi}/\zeta_{.}$. For Case II for the direct key, $E(N_{\alpha b}|$ is estimated by $\hat{B}_{\alpha}\zeta_{\alpha} / (q_{(\alpha)b}\,\delta g)$, where $q_{(\alpha)\beta}$ is the mean value at t of q_i for the $N_{\alpha\beta}$ fish which can contribute to $c_{\alpha\beta}$. Hence $E(B_{\alpha}| / E^{\frac{1}{2}}(N_{\alpha b}|$ is estimated by $(\hat{B}_{\alpha} q_{(\alpha)b}\,\delta g / \zeta_{\alpha})^{\frac{1}{2}}$. $q_{(\alpha)b}$ must be estimated in some manner from $\hat{q}$. For instance, it might be reasoned that $q_{(\alpha)b}$ is independent of α . Similarly, for Case II for the inverse key, $E(N_{\ell m}|$ is estimated by $\hat{\gamma}_{\ell m}(\hat{\gamma}^{-1}_{**}\zeta_*)_m / (q_{(\ell)m}\delta g)$, where $\hat{\gamma}_{**}$ estimates γ_{**} .
$\therefore$ $E(\gamma_{\ell m}| / E^{\frac{1}{2}}(N_{\ell m}|$ is estimated by
$\{\hat{\gamma}_{\ell m} q_{(\ell)m}\delta g / (\hat{\gamma}^{-1}_{**}\zeta_*)_m\}^{\frac{1}{2}}$. Likewise,

$E(\gamma_{\ell[X_{m(*)},S_b]}| / E^{\frac{1}{2}}(N_{\ell[X_{m(*)},S_b]}|$ is estimated by

$\left\{\hat{\gamma}_{\ell[X_{m(*)},S_b]}\, q_{(\ell)[X_{m(*)},S_b]}\,\delta g \,/\, (\hat{\gamma}^{-1}_{*[*,S_b]}\,\zeta_*)_m\right\}^{\frac{1}{2}}$ and

$E(\gamma_{\ell[X_b,S_m]}| / E^{\frac{1}{2}}(N_{\ell[X_b,S_m]}|$ is estimated by

$\left\{\hat{\gamma}_{\ell[X_b,S_m]}\, q_{(\ell)[X_b,S_m]}\,\delta g \,/\, (\hat{\gamma}^{-1}_{*[X_b,*]}\,\zeta_*)_m\right\}^{\frac{1}{2}}$. For Case III,

calculation of standard deviations of key matrix elements has been discussed in Section V(b)(i).

If conditions (1), (2) or (3) are violated, the remedial actions of Table 4 can be attempted.

TABLE 4

REMEDIAL ACTIONS TO CORRECT UNFULFILLED CONDITIONS

Unfulfilled condition	Remedial action			
	(a)	(b)	(c)	(d)
(1) $q\,\delta g + M\,\delta t \ll 1$	X	—	—	—
(2) $E(\mu| \gg 1$	X	X	—	—
(3) key deterministic	—	X	X	X

(a) Redefine sampling interval δt .

(b) Redefine cohort interval Δt .

(c) Redefine size class intervals (Cases II, III).

(d) Group cohorts for which ξ not to be calculated (Cases II, III - inverse key).

Example:—

The condition $q\,\delta g + M\,\delta t \ll 1$ can be taken as the region (of the first quadrant) of Figure 8 to the left of the line $\hat{q}\,\delta g + \hat{M}\,\delta t = \cdot 1$.

Assume $E(\mu|$ is proportional to δt and also to the cohort interval for cohort b (i.e. Δt). Then the condition $E(\mu| \gg 1$ can be taken as the region above the curve $\frac{\Delta t\,\delta t}{\Delta t_1\,\delta t_1} = \frac{100}{\hat{\xi}_1}$, where

$\hat{\xi}_1$ is the value of $\hat{\xi}$ corresponding to Δt_1 , δt_1 .

Hence the shaded area of Figure 8 represents the domain of δt , Δt in which the conditions (1) and (2) are both satisfied.

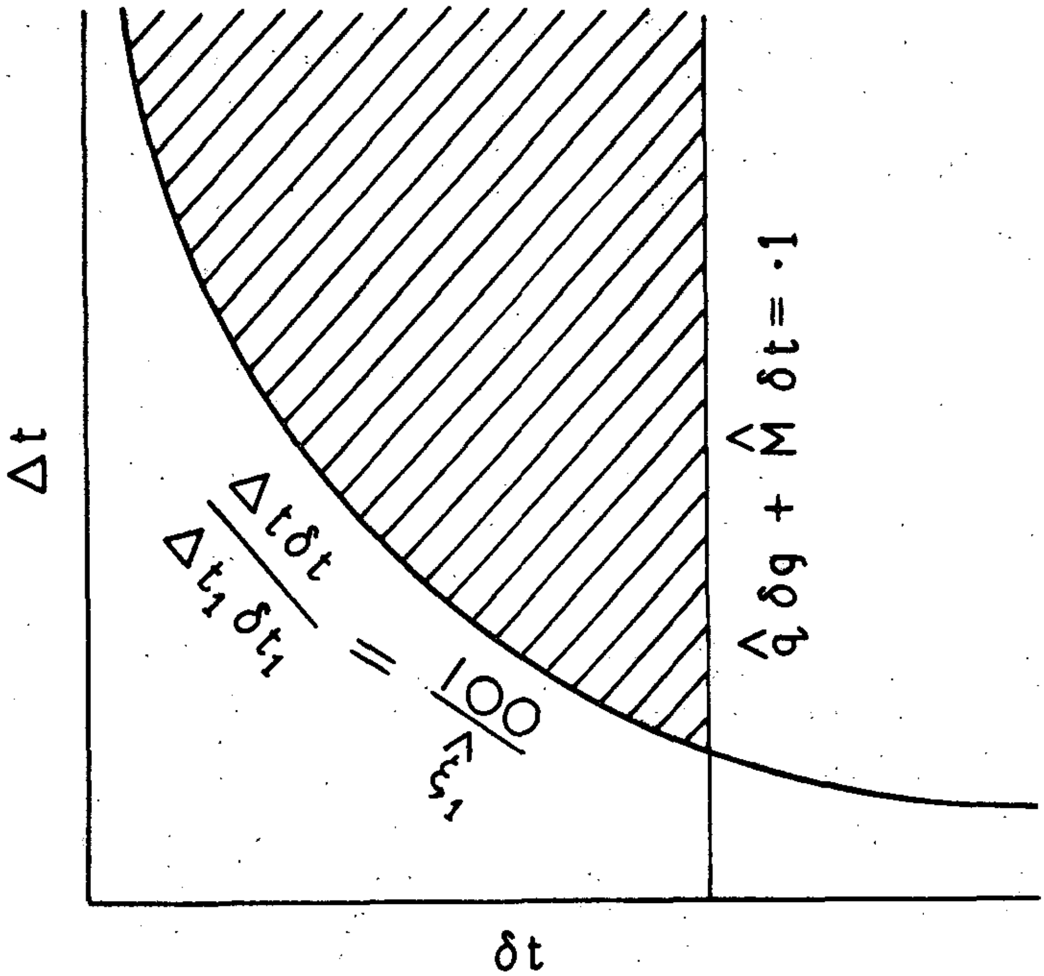

Fig. 8. Domain (shaded area) of δt , Δt in which the conditions (1) and (2) of Table 4 are both satisfied.

X. OPTIMUM LENGTH OF SAMPLING INTERVALS, COHORT INTERVALS, AND SIZE CLASS INTERVALS

Given that the above conditions (1), (2), (3) are fulfilled, what are the optimum lengths of sampling intervals, cohort intervals and size class intervals?

The $\tau^2(\hat{L})$ will depend upon the sizes of these intervals. In addition, the following points are noted:

(i) *Sampling Intervals*

Smaller sampling intervals will give more information on q, M for a given cohort as functions of time.

(ii) *Cohort Intervals*

Smaller cohort intervals will give more information on q, M at a given time as functions of cohort age.

(iii) *Size Class Intervals (for Cases II, III)*

The accuracy to which the size of a fish can be measured sets a natural lower limit to the length of the size class intervals. The following example shows that making the size class intervals this small will not necessarily minimise the $\tau^2(\hat{L})$.

Example: –

Suppose Λ consists of two size classes H_1, H_2 and two cohorts (of the same sex) with ages X_1, X_2 at t. Suppose s is found to contain $c_{1\cdot}$ in H_1 and $c_{2\cdot}$ in H_2. Assuming the $p_{\alpha\beta}$ $(= \mu_{\alpha\beta}/\mu_{\cdot\cdot})$ to be known exactly, $\mu_{\cdot 1}$ can be estimated either by ξ_1 or by ξ_2, defined as follows:

$$\xi_1 \equiv (p_{11}+p_{21})(c_{1\cdot}+c_{2\cdot}) = \frac{\mu_{\cdot 1}}{\mu_{\cdot\cdot}}\, c_{\cdot\cdot} \quad \text{does not use the}$$

cohort/size key.

$$\xi_2 \equiv \frac{p_{11}}{p_{11}+p_{12}}\, c_{1\cdot} + \frac{p_{21}}{p_{21}+p_{22}}\, c_{2\cdot} = B_1 c_{1\cdot} + B_2 c_{2\cdot} \quad , \text{ where}$$

$B_1 \equiv \dfrac{\mu_{11}}{\mu_{1\cdot}}$, $B_2 \equiv \dfrac{\mu_{21}}{\mu_{2\cdot}}$. ξ_2 uses the cohort/size key.

Both are unbiased:

$$E(\xi_1|\Lambda) = \frac{\mu_{\cdot 1}}{\mu_{\cdot\cdot}}\, \mu_{\cdot\cdot} = \mu_{\cdot 1} \quad .$$

$$E(\xi_2|\Lambda) = \frac{\mu_{11}}{\mu_{1\cdot}}\, \mu_{1\cdot} + \frac{\mu_{21}}{\mu_{2\cdot}}\, \mu_{2\cdot} = \mu_{11}+\mu_{21} = \mu_{\cdot 1} \quad .$$

Which has the smaller variance ?:

$$\sigma^2(\xi_2|\Lambda) - \sigma^2(\xi_1|\Lambda) = B_1^2\mu_{1\cdot} + B_2^2\mu_{2\cdot} - (\frac{\mu_{\cdot 1}}{\mu_{\cdot\cdot}})^2 \mu_{\cdot\cdot}$$

$$= B_1^2\mu_{1\cdot} + B_2^2\mu_{2\cdot} - (B_1\mu_{1\cdot} + B_2\mu_{2\cdot})^2/\mu_{\cdot\cdot}$$

$$= \frac{\mu_{1\cdot}\mu_{2\cdot}}{\mu_{\cdot\cdot}}(B_1 - B_2)^2 \geq 0 \quad .$$

So, in this example, using the key has increased the variance of ξ unless $B_1 = B_2$. However, the example is unrealistic, since in practice the $p_{\alpha\beta}$ are not known exactly and sampling s for size supplies more information. But there will be no more additional information after the size class intervals are shortened past a certain point, for $CV(c_{\alpha\cdot}|\Lambda)$ increases as $\mu_{\alpha\cdot}^{-\frac{1}{2}}$.

XI. GENERAL SOLUTION OF THE EQUATIONS

$$\log\left(\frac{\hat{q}'}{\hat{q}''}\right) + \int_{Q'}^{Q''} \hat{q}\, dg + \hat{M}\, dt = \hat{L}$$

The time pairs for a cohort are chosen, without loss of generality, by pairing successive sampling intervals for that cohort. For, from the resulting equations, the equations for any other choice of time pairs can be derived.

For this numerical analysis it is convenient to consider the whole time axis partitioned into intervals δt of which the sampling intervals are a subset. Denote the δt by δt_1 , δt_2 , ... from the first sampling interval δt_1 . For δt_i , let δg_{Di} be the effort in zone $D\,(D = 1,2, \ldots)$ and let q_{bi} , M_{bi} be the values of q , M for cohort b at the midpoint $t_{(i)}$.

Let cohort b be in zone D(b).

Let Q_i be the point on the curve of Figure 1 corresponding to $t_{(i)}$.

Suppose δt_i , δt_j are successive sampling intervals for cohort b. The function $\log\left(\frac{q_{bi}}{q_{bj}}\right) + \int_{Q_i}^{Q_j} q\, dg + M\, dt$, the value of which is L_{bij} say,
cohort b

is replaced by the function ℓ_{bij} of $q_{bi}, q_{b,i+1}, \ldots, q_{bj}, M_{bi}, M_{b,i+1}, \ldots, M_{bj}$, which is defined as

$$\ell_{bij} \equiv \log\left(\frac{q_{bi}}{q_{bj}}\right)$$

$$+ \frac{\delta g_{D(b)i}}{2}\left(q_{bi} + \frac{\delta t_i/4}{\delta t_i/2 + \delta t_{i+1}/2}\left[q_{b,i+1} - q_{bi}\right]\right)$$

$$+ \delta g_{D(b),i+1}\, q_{b,i+1} + \ldots + \delta g_{D(b),j-1}\, q_{b,j-1}$$

$$+ \frac{\delta g_{D(b)j}}{2}\left(q_{bj} - \frac{\delta t_j/4}{\delta t_j/2 + \delta t_{j-1}/2}\left[q_{bj} - q_{b,j-1}\right]\right)$$

$$+ \frac{\delta t_i}{2}\left(M_{bi} + \frac{\delta t_i/4}{\delta t_i/2 + \delta t_{i+1}/2}\left[M_{b,i+1} - M_{bi}\right]\right)$$

$$+ \delta t_{i+1} M_{b,i+1} + \ldots + \delta t_{j-1} M_{b,j-1}$$

$$+ \frac{\delta t_j}{2}\left(M_{bj} - \frac{\delta t_j/4}{\delta t_j/2 + \delta t_{j-1}/2}\left[M_{bj} - M_{b,j-1}\right]\right)$$

$$= \log\left(\frac{q_{bi}}{q_{bj}}\right) + \sum_{k=i}^{j}\left[\delta G_{D(b)ijk}\, q_{bk} + \delta T_{ijk} M_{bk}\right]$$

, say. (Note that subscript elements of more than one digit, or more than one term, are separated by commas.)

Even if the values L_{bij} were known exactly, the equations $\ell_{bij} = L_{bij}$ could not be solved uniquely, because there are too many unknowns q_{bi}, M_{bi}. So assumptions must be made on the functional form of the q and M values for the population in question. This in effect introduces parameters $z_* = (z_1, z_2, \ldots, z_m)$ such that $q_{bi} = q_{bi}(z_*)$, $M_{bi} = M_{bi}(z_*)$. (As an example, the functional form of q and M for rock lobster fisheries will be discussed later.) On replacing the q_{bk}, M_{bk} by the functions $q_{bk}(z_*)$, $M_{bk}(z_*)$ in the ℓ_{bij}, it is assumed that there are now many more equations than unknowns $z_1, \ldots, z_m$. So these equations will contain redundancies.

However, if the L_{bij} are replaced by their estimates, the $\hat{L}_{bij}$, then these redundancies are not only removed but, what is more, the equations become inconsistent. The solution $\hat{z}_* = (\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_m)$ suggested here is the best fit solution in the least squares sense and is defined as follows:

Let $\hat{q}_{bk}(\hat{z}_*)$, $\hat{M}_{bk}(\hat{z}_*)$ be of the same functional form as q_{bk} , M_{bk} but with $\hat{z}_*$ replacing z_* . Let $\hat{\ell}_{bij}$ be of the same functional form as ℓ_{bij} but with $\hat{q}_{bk}$, $\hat{M}_{bk}$ replacing q_{bk} , M_{bk} . Then $\hat{z}_*$ is found by minimising $\sum_{(bij)} (\hat{\ell}_{bij} - \hat{L}_{bij})^2$ with respect to $\hat{z}_*$, i.e. by solving $\frac{\partial}{\partial \hat{z}_\ell} \sum_{(bij)} (\hat{\ell}_{bij} - \hat{L}_{bij})^2 = 0$, $\ell = 1, \ldots, m$. This system reduces to

$$\sum_{(bij)} \left\{ \left(\log\left[\frac{\hat{q}_{bi}}{\hat{q}_{bj}}\right] + \sum_{k=i}^{j} \left[\delta G_{D(b)ijk}\, \hat{q}_{bk} + \delta T_{ijk} \hat{M}_{bk}\right] - \hat{L}_{bij} \right) \left(\frac{\partial \hat{q}_{bi}}{\partial \hat{z}_\ell} / \hat{q}_{bi} - \frac{\partial \hat{q}_{bj}}{\partial \hat{z}_\ell} / \hat{q}_{bj} + \sum_{k=i}^{j} \left[\delta G_{D(b)ijk} \frac{\partial \hat{q}_{bk}}{\partial \hat{z}_\ell} + \delta T_{ijk} \frac{\partial \hat{M}_{bk}}{\partial \hat{z}_\ell}\right] \right) \right\} = 0 ,$$

$\ell = 1, \ldots, m$. This is a system of m equations in the m unknowns $\hat{z}_1, \ldots, \hat{z}_m$. Having solved for $\hat{z}_*$, then the estimates of the q_{bk} and the M_{bk} , i.e. the $\hat{q}_{bk}$ and the $\hat{M}_{bk}$, can be calculated.

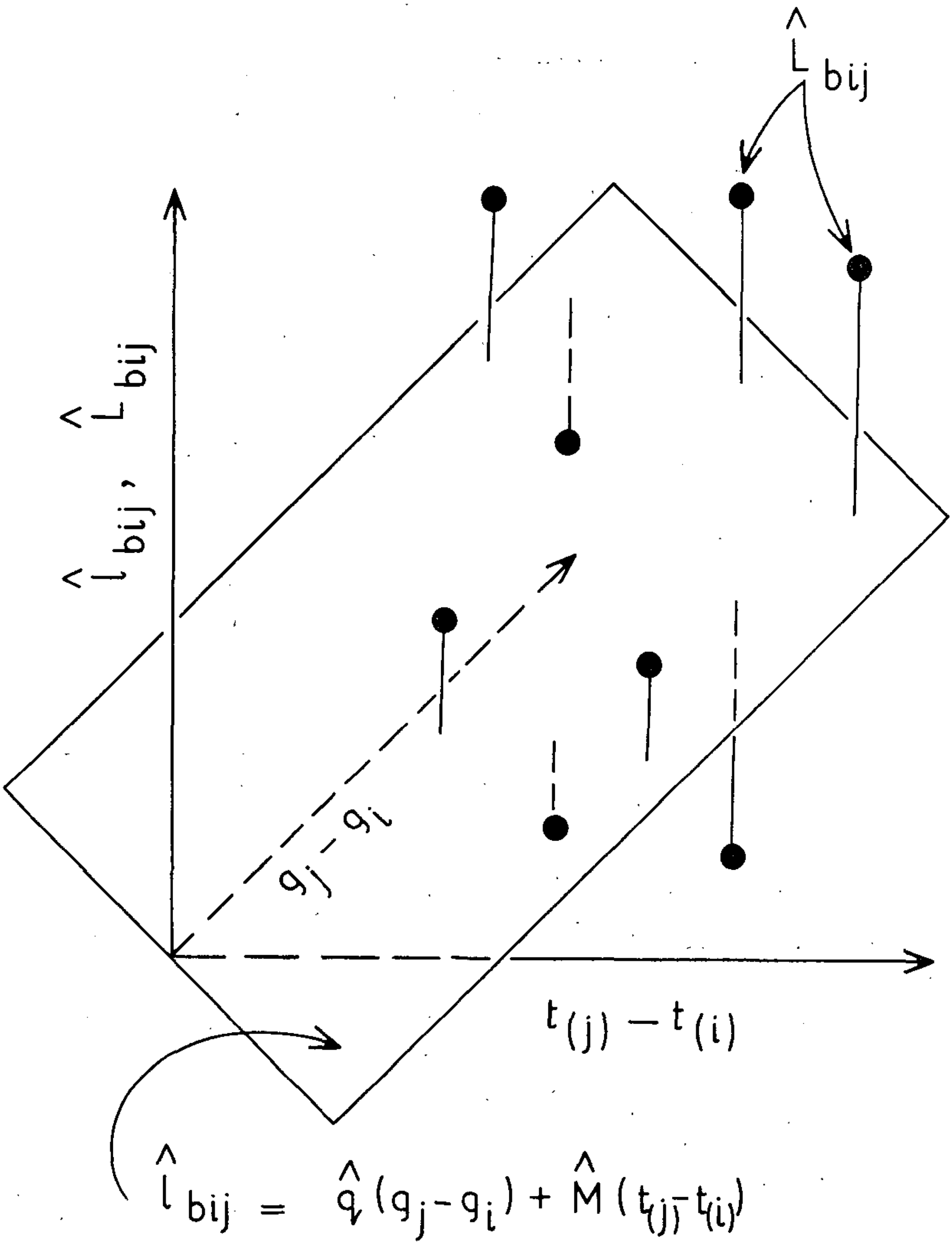

Fig. 9. Illustrating method for finding $\hat{q}$, $\hat{M}$ when q_{bk}, M_{bk} are constants.

Example (1): –

Let q_{bk} = constant = q , say, all b , k ; M_{bk} = constant = M , say, all b , k . So m = 2 , $z_1 = q$, $z_2 = M$. Hence $\hat{q}_{bk} = \hat{q}$, say, all b , k and $\hat{M}_{bk} = \hat{M}$, say, all b , k . Then $\hat{\ell}_{bij} = \hat{q}(g_j - g_i) + \hat{M}(t_{(j)} - t_{(i)})$, the equation of a plane (Fig. 9), where $g_k \equiv g(t_{(k)})$.

Minimising $\sum_{(bij)}\left(\hat{\ell}_{bij} - \hat{L}_{bij}\right)^2$ with respect to $\hat{q}$, $\hat{M}$ gives

$$\sum_{(bij)}[(g_j - g_i)\hat{q} + (t_{(j)} - t_{(i)})\hat{M} - \hat{L}_{bij}][g_j - g_i] = 0 ,$$

$$\sum_{(bij)}[(g_j - g_i)\hat{q} + (t_{(j)} - t_{(i)})\hat{M} - \hat{L}_{bij}][t_{(j)} - t_{(i)}] = 0 .$$ The $\hat{q}$, $\hat{M}$ satisfying these two equations give the plane of best fit to points $\hat{L}_{bij}$.

If also g' exists and is constant $\left(\text{equal to } \dfrac{g_j - g_i}{t_{(j)} - t_{(i)}}\right)$, then $\hat{\ell}_{bij} = \hat{Z}(t_{(j)} - t_{(i)})$, the equation of a line, where $\hat{Z} \equiv \hat{q}g' + \hat{M}$. The above two equations for $\hat{q}$, $\hat{M}$ both reduce to

$$\sum_{(bij)}[\hat{Z}(t_{(j)} - t_{(i)}) - \hat{L}_{bij}][t_{(j)} - t_{(i)}] = 0 .$$ The $\hat{Z}$ satisfying this equation gives the least squares regression line fitted to the points $\hat{L}_{bij}$. Gulland (1969) describes essentially this solution for this simple case, but instead of $\hat{L}$ uses essentially the more biased $\log\left(\dfrac{\hat{\xi}'/\delta g'}{\hat{\xi}''/\delta g''}\right)$.

Example (2): –

The purpose of this example is to show how the method of Murphy(1965) can be regarded as a special case of the general model described herein.

First note that if g' exists, except perhaps at isolated points, then the substitution $F = q\,g'$ can be made in the basic equation $L = \log\left\{\frac{E(\xi'/\delta g'|}{E(\xi''/\delta g''|}\right\}$ of Section IV. This substitution eliminates the effort explicitly, but information on the functional form of mortality rates is lost. The resulting transformed equation is $\log\left(\frac{F'}{F''}\right) + \int_{Q'}^{Q''} Z\,dt = \log\left\{\frac{E(\xi'/\delta t'|}{E(\xi''/\delta t''|}\right\}$, where F' and F'' are the values of F at t' , t'' respectively.

Murphy made just sufficient assumptions to solve for mortality rates without using the least squares method. Also, he considered the catch over an extended (unit) time period - cf. the requirement of Section IX that for a sampling interval $q\,\delta g + M\delta t \ll 1$.

Consider cohort b only. Let M = constant. Within a certain unit time period, which is here divided into the sampling intervals $\delta t_\ell , \ldots , \delta t_m$, assume g' exists and F = constant = F_1 , say. Likewise, within the next unit time period, which is here divided into the sampling intervals $\delta t_{m+1} , \ldots , \delta t_n$, assume g' exists and F = constant = F_2 , say. Let $Z_1 \equiv F_1 + M$, $Z_2 \equiv F_2 + M$.

The least squares method can be avoided by combining the transformed equations for the various time pairs in the following manner before solving:

Let $\xi_{(i)}$, $\hat{\xi}_{(i)}$ be respectively the estimators ξ , $\hat{\xi}$ for δt_i . Let

$$\xi_1 \equiv \sum_{i=\ell}^{m} \xi_{(i)} , \quad \hat{\xi}_1 \equiv \sum_{i=\ell}^{m} \hat{\xi}_{(i)} , \quad \xi_2 \equiv \sum_{i=m+1}^{n} \xi_{(i)} , \quad \hat{\xi}_2 \equiv \sum_{i=m+1}^{n} \hat{\xi}_{(i)} .$$

When δt_i is in the first unit time period, the above transformed equation corresponding to time pair $t_{(\ell)}$, $t_{(i)}$ can be written $\frac{E(\xi_{(i)}|}{E(\xi_{(\ell)}|} = \frac{\delta t_i}{\delta t_\ell}\, e^{-Z_1(t_{(i)} - t_{(\ell)})}$. Adding (integrating) such equations for $i = \ell, \ldots, m$ gives $E(\xi_1| = \frac{E(\xi_{(\ell)}|}{\delta t_\ell Z_1} e^{Z_1 \frac{\delta t_\ell}{2}} (1 - e^{-Z_1})$.

Likewise, $E(\xi_2| = \frac{E(\xi_{(m+1)}|}{\delta t_{m+1} Z_2} e^{Z_2 \frac{\delta t_{m+1}}{2}} (1 - e^{-Z_2})$.

Also, $\frac{E(\xi_{(m+1)}|}{E(\xi_{(\ell)}|} = \frac{F_2}{F_1} \frac{\delta t_{m+1}}{\delta t_\ell} e^{-Z_1 + Z_1 \frac{\delta t_\ell}{2} - Z_2 \frac{\delta t_{m+1}}{2}}$.

$$\therefore \frac{E(\xi_2|}{E(\xi_1|} = \frac{\frac{F_2}{Z_2}(1 - e^{-Z_2})}{\frac{F_1}{Z_1}(1 - e^{-Z_1})} e^{-Z_1} .$$

Relationships of the form $\frac{\hat{\xi}_2}{\hat{\xi}_1} = \frac{\frac{\hat{F}_2}{\hat{Z}_2}(1 - e^{-\hat{Z}_2})}{\frac{\hat{F}_1}{\hat{Z}_1}(1 - e^{-\hat{Z}_1})} e^{-\hat{Z}_1}$ were used by Murphy (1965), where $\hat{F}_i$, $\hat{Z}_i$ estimate F_i , Z_i respectively.

If the $\hat{\xi}_{(i)}$ are not combined as above, then the (transformed) basic equations will be inconsistent and the least squares method is appropriate. This method minimizes

$$\sum_i \left(\log\left[\frac{\hat{F}_{bi}}{\hat{F}_{b,i+1}}\right] + \hat{Z}_{bi} \frac{\delta t_i}{2} + \hat{Z}_{b,i+1} \frac{\delta t_{i+1}}{2} - \hat{L}'_{bi,i+1} \right)^2 , \text{ where}$$

$\hat{F}_{bi}$, $\hat{Z}_{bi}$ estimate F , Z respectively for $t_{(i)}$, and $\hat{L}'_{bij}$ has the same form as $\hat{L}_{bij}$ (see Section VII(iii)) but with δt_i , δt_j replacing $\delta g_{D(b)i}$, $\delta g_{D(b)j}$ respectively.

It is seen that combining the $\hat{\xi}_{(i)}$ results in a loss of data, for $i = \ell, \ldots, m-1$ in the least squares sum is sufficient to solve for $\hat{Z}_1$.

[Also, as noted in Section X, smaller sampling intervals allow more detailed information to be obtained on q , M for a given cohort as functions of time.]

XII. CALCULATION OF CONFIDENCE RANGES OF THE q_{bk} AND THE M_{bk}

First the goodness of fit of the assumptions on the functional form of q , M must be tested. If the fit is satisfactory, then the confidence ranges can be calculated.

(i) *Testing Goodness of Fit*

Let there be n cohort-time pairs (bij) and let m be the order of z_* . Assume that, approximately, each $\hat{L}_{bij} \frown$ normal $\left\{\text{mean } L_{bij} \text{, variance } \tau^2(\hat{L}_{bij})\right\}$ and that the variates $\hat{L}_{bij}$ are independent. Divide the possible values of $\hat{L}_{bij}$ into m+2 ranges of equal probability. Let O_k be the actual number of the n cohort-time pairs for which $\hat{L}_{bij}$ falls in its k^{th} range. Then

$$\sum_{k=1}^{m+2} \frac{(O_k - \frac{n}{m+2})^2}{n/(m+2)} \frown \chi^2(m+1) ,$$ where m+1 is the number of degrees of freedom.

Now to find the ranges of equal probability the estimates $\hat{\ell}_{bij}$ are used. These estimates are based on the parameter estimates $\hat{z}_1 , \ldots , \hat{z}_m$. Let O'_* be the value of O_* for the estimated ranges.

Then $\sum_{k=1}^{m} \frac{(O'_k - \frac{n}{m+2})^2}{n/(m+2)} \frown \chi^2(1)$. So, as

$Pr[\chi^2(1) \geq 3 \cdot 8] \simeq 0 \cdot 05$, the data are compatible with the assumptions on the functional form of q , M if

$$\sum_{k=1}^{m} \frac{(O'_k - \frac{n}{m+2})^2}{n/(m+2)} < 3 \cdot 8 \quad .$$

(The above χ^2 approximation is satisfactory provided $\frac{n}{m+2} \geq 5$, i.e. number of equations $\geq 5 \times$ number of unknowns + 10 .)

(ii) *Calculation of Confidence Ranges*

On the assumption that each $\hat{L}_{bij} \frown$ normal $\{$mean L_{bij}, variance $\tau^2(\hat{L}_{bij})\}$, the joint frequency function $f(\hat{L}_{***})$ is a known function of the L_{bij} and the $\tau^2(\hat{L}_{bij})$.

$$\tau^2(\hat{q}_{bk}) \equiv E(\hat{q}_{bk} - q_{bk})^2 = \int f(\hat{L}'_{***})(\hat{q}'_{bk} - q_{bk})^2 \, d\hat{L}'_{***} \ .$$

This integral is evaluated by the method of Appendix VI. For each $\hat{L}'_{***}$ selected $\sum_{(bij)} (\hat{\ell}_{bij} - \hat{L}'_{bij})^2$ is minimised to find $\hat{q}'_{bk}$. q_{bk} and $f(\hat{L}'_{***})$ will depend upon L_{***} which is not known. So $\hat{L}_{***}$ is used instead of L_{***} and $\hat{q}_{bk}$ instead of q_{bk} .

Then, by Appendix IX,
$\Pr\{|\hat{q}_{bk} - q_{bk}| < (\eta^2 + 1)^{\frac{1}{2}} \tau(\hat{q}_{bk})\} \geq \Pr\{|\hat{q}_{bk} - E(\hat{q}_{bk})| < \eta\, \sigma(\hat{q}_{bk})\}$.
So, if $\hat{q}_{bk} \frown$ normal (which is assumed to hold approximately), then with at least 95% confidence
$|\hat{q}_{bk} - q_{bk}| < (1{\cdot}96^2 + 1)^{\frac{1}{2}} \tau(\hat{q}_{bk}) = 2{\cdot}2\, \tau(\hat{q}_{bk})$.

Similarly for $\hat{M}_{bk}$.

XIII. FUNCTIONAL FORM OF q AND M FOR ROCK LOBSTER FISHERIES

It is not claimed here that the following assumptions hold for any definite rock lobster fishery. The picture presented merely represents a plausible starting point for the application of the model to such a fishery to estimate q and M values. In any particular application, each assumption should be investigated and either verified or modified.

The lobsters are caught in pots. The unit of effort is one "potlift" which consists of laying a baited pot and retrieving it some time later. It is assumed that a potlift has an area of effect, A_e, within which a lobster has a probability p of being caught by that potlift. The potlift catches no lobsters outside A_e. Assume A_e to be constant for all potlifts.

Assume that fishing is only on discrete grounds (associated with the reefs where the lobsters live). Assume that the fishing operations on a ground consist of a sequence of "trips", a trip consisting of a number of pots laid and lifted (by one or more boats) more or less simultaneously. Assume that in a trip the pots are laid randomly over the fishing ground area and that there is negligible overlap of the A_e.

Assume that when the cohort interval of a lobster is recruited that the lobster is on a fishing ground from which it will not migrate. This assumption is supported by tagging experiments on *Jasus novaehollandiae* Holthuis (Fielder and Olsen 1967) and on *Jasus lalandii* (H. Milne Edwards) (Heydorn 1969). Note that the species of the genus *Jasus* recognised in this paper are described by George and Kensler (1970).

Consider the entire fishery to be partitioned into zones, each zone containing one or more whole fishing grounds (Fig. 10). A zone is the smallest area for which regular effort statistics are collected.

Let each cohort interval be one year long. Let the birthdate of each cohort be on the same day of the year, so cohort age specifies season.

Call a cohort each subset of those members of R that have the same cohort interval, are of the same sex, and inhabit the same zone from the date that the cohort interval is recruited.

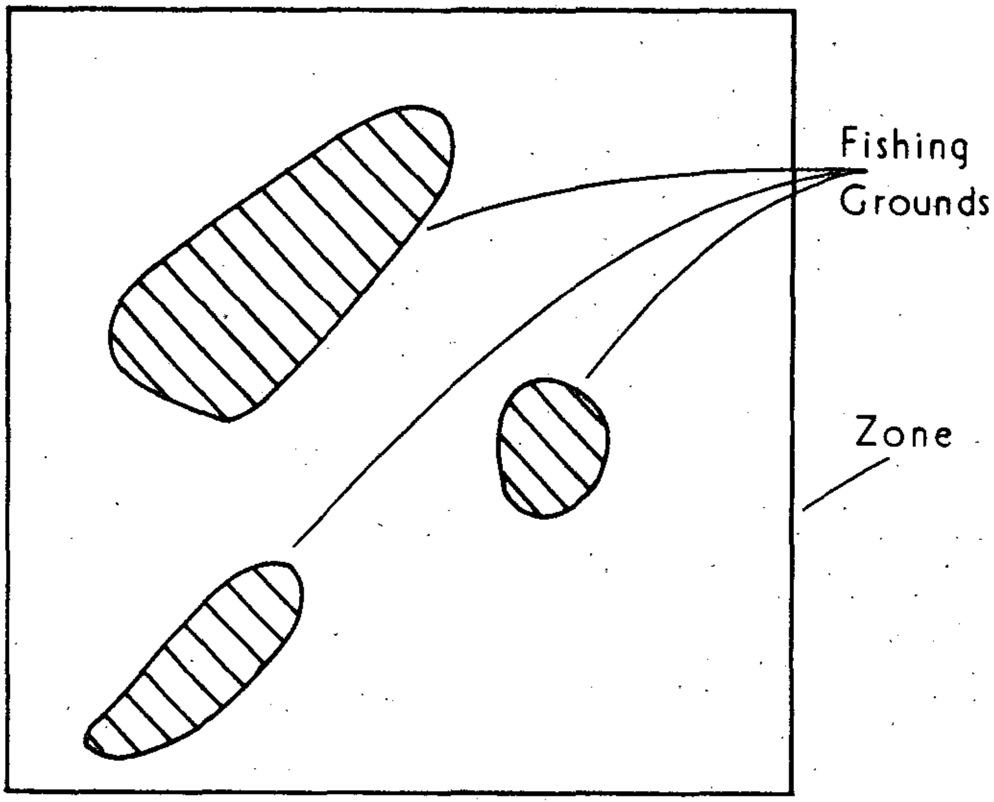

Fig. 10. Illustrating assumed structure of a rock lobster fishery.

Consider the i^{th} lobster of a cohort in a zone of total fishing ground area A, which lives on a fishing ground of area A_G. Suppose time interval dt consists of subintervals dt_j, $j = 1, 2, \ldots$ during each of which there is just one trip to A_G. Suppose the j^{th} trip contains dg_{Gj} potlifts. Then

Pr(lobster i survives trip j| alive at beginning of dt_j) =

$$1 - \frac{A_e p_i}{A_G} dg_{Gj} - M_i dt_j \text{ , where } p_i \text{ is p for lobster i .}$$

∴ Pr(lobster i survives dt| alive at beginning of dt)

$$= \prod_j \left(1 - \frac{A_e p_i}{A_G} dg_{Gj} - M_i dt_j\right) .$$

∴ Pr(lobster i dies during dt| alive at beginning of dt)

$$= 1 - \prod_j \left(1 - \frac{A_e p_i}{A_G} dg_{Gj} - M_i dt_j\right)$$

$$= \frac{A_e p_i}{A_G} dg_G + M_i dt + \ldots , \quad \text{where } dg_G = \sum_j dg_{Gj} .$$ The higher order terms are neglected. Assume the potlifts are being distributed randomly over the entire fishing ground area of the zone so $dg_G / A_G = dg / A$, where dg is the number of potlifts in the zone during dt. Then

Pr(lobster i dies during dt| alive at beginning of dt) = $q_i dg + M_i dt$,

where $q_i = \frac{A_e p_i}{A}$.

It is reasonable to assume that the probability of a lobster entering a pot to eat the bait will depend upon the natural food available, and that the latter will depend upon the locality. So assume p_i depends upon zone.

Hickman (1946), working on *Jasus novaehollandiae* Holthuis, and Heydorn (1969) report an annual biological cycle involving reproduction and moulting. Feeding activity is related to this cycle; for instance, much fewer lobsters enter the pots during the mating and moulting seasons. What is more, the cycle is different for males and females; for instance, the different sexes moult at different times of the year. Hence it is assumed that p_i depends upon the sex of lobster i, and the season. A seasonal restriction on catching one sex (i.e. no berried females to be taken, or a closed season for one sex) would have a similar effect on p_i.

p_i depends also upon the size of lobster i. However, it is assumed that p_i is independent of size once the lobster is larger than a certain size (related to the size of first capture). Such a lobster is called "full sized".

Allow only "full sized" cohorts (i.e. with virtually all members full sized) to enter the equations to be solved for $\hat{q}$, $\hat{M}$. For such a cohort, then, it is assumed that q depends (through the p_i) upon the sex, season and zone. To reduce the number of unknowns still further, assume that q is of the form $\alpha\beta$, where α depends upon sex (S) and season, and β upon zone. The average of the β over the zones is arbitrarily made unity (expressed as $\bar{\beta} = 1$).

Hickman (1945) and Heydorn (1969) note that the exoskeleton is soft at the moulting period. So it is plausible to assume that the lobsters are more susceptible to predation during this period. Thus Heydorn (1969) remarks that hagfishes or dogfish have only been seen attacking damaged or moulting rock lobsters. As ecdysis occurs at a definite time in the annual biological cycle, assume that M_i depends upon the sex of lobster i, and the season.

It is also plausible to assume that younger lobsters are less able to defend themselves than older ones and are therefore more susceptible to predation. Thus Heydorn (1969) remarks that rockfish appear to be capable of inflicting considerable damage on the early adult stages. Hence assume M_i depends upon the age of lobster i.

So, as cohort age specifies season, it is assumed that M for the cohort depends upon the sex and the cohort age (X).

To reduce the number of unknowns still further, write α as a Fourier expansion in t, and assume M can be written as the sum of a Fourier expansion and a polynomial in X. So write

$$\alpha = \alpha_S(t) \equiv \alpha_{S0} + \sum_{\eta=1}^{\ell} \alpha_{S\eta} \sin(2\pi\eta t) + \alpha'_{S\eta} \cos(2\pi\eta t) \quad \text{and}$$

$$M = m_S(X) \equiv \sum_{\eta=1}^{m} m_{S\eta} \sin(2\pi\eta X) + m'_{S\eta} \cos(2\pi\eta X) + \sum_{r=0}^{n} m''_{Sr} X^r ,$$

where t and X are measured in years. Hence

$z_* = \alpha_{S0}, \alpha_{S\eta}, \alpha'_{S\eta}$ $(S = 1,2\,;\ \eta = 1, \ldots, \ell)$;

β_D (the value of β, for various zones D) ;

$m_{S\eta}, m'_{S\eta}, m''_{Sr}$ $(S = 1,2\,;\ \eta = 1, \ldots, m\,;\ r = 0, \ldots, n)$.

XIV. THE EQUATIONS $\log\left(\frac{\hat{q}'}{\hat{q}''}\right) + \int_{Q'}^{Q''} \hat{q}\,dg + \hat{M}\,dt = \hat{L}$ FOR ROCK LOBSTER FISHERIES

With the above assumptions on the functional form of q and M, the least squares method reduces to solving the following equations for $\hat{z}_* \equiv \hat{\alpha}_{**}, \hat{\alpha}'_{**}, \hat{\beta}_*, \hat{m}_{**}, \hat{m}'_{**}, \hat{m}''_{**}$. In writing these equations, the abbreviation

$$F_{bij} \equiv \log\left[\frac{\hat{\alpha}_{S_b}(t_{(i)})}{\hat{\alpha}_{S_b}(t_{(j)})}\right] + \sum_{k=i}^{j}\left[\delta G_{D(b)ijk}\, \hat{\alpha}_{S_b}(t_{(k)})\, \hat{\beta}_{D(b)} + \delta T_{ijk}\, \hat{m}_{S_b}(X_{bk})\right] - \hat{L}_{bij}$$

is used, where $\hat{\alpha}_S(t)$, $\hat{m}_S(X)$ are of the same functional form as $\alpha_S(t)$, $m_S(X)$ but with $\hat{z}_*$ replacing z_*, and X_{bi} is the age of cohort b at $t_{(i)}$. Also, the function $\delta(y,z) \equiv 1,0$ if $y = z$, $y \neq z$. ℓ, m, n are as in Section XIII. Then:

$$\sum_{(bij)} \delta(S_b, S)\, F_{bij}\left\{\frac{1}{\hat{\alpha}_{S_b}(t_{(i)})} - \frac{1}{\hat{\alpha}_{S_b}(t_{(j)})} + \sum_{k=i}^{j} \delta G_{D(b)ijk}\, \hat{\beta}_{D(b)}\right\} = 0, \quad S = 1,2 .$$

$$\sum_{(bij)} \delta(S_b, S)\, F_{bij}\left\{\frac{\sin(2\pi\eta t_{(i)})}{\hat{\alpha}_{S_b}(t_{(i)})} - \frac{\sin(2\pi\eta t_{(j)})}{\hat{\alpha}_{S_b}(t_{(j)})} + \sum_{k=i}^{j} \delta G_{D(b)ijk} \sin(2\pi\eta t_{(k)})\, \hat{\beta}_{D(b)}\right\} = 0,$$

$$S = 1, 2;\ \eta = 1, \ldots, \ell .$$

$$\sum_{(bij)} \delta(S_b, S)\, F_{bij}\left\{\frac{\cos(2\pi\eta t_{(i)})}{\hat{\alpha}_{S_b}(t_{(i)})} - \frac{\cos(2\pi\eta t_{(j)})}{\hat{\alpha}_{S_b}(t_{(j)})} + \sum_{k=i}^{j} \delta G_{D(b)ijk} \cos(2\pi\eta t_{(k)})\, \hat{\beta}_{D(b)}\right\} = 0,$$

$$S = 1, 2;\ \eta = 1, \ldots, \ell .$$

$$\sum_{(bij)} \delta(D(b), D)\, F_{bij}\left\{\sum_{k=i}^{j} \delta G_{D(b)ijk}\, \hat{\alpha}_{S_b}(t_{(k)})\right\} = 0, \quad D = 1, 2, \ldots .$$

$$\sum_{(bij)} \delta(S_b, S) F_{bij} \left\{ \sum_{k=i}^{j} \delta T_{ijk} \sin(2\pi\eta X_{bk}) \right\} = 0 ,$$
$$S = 1, 2 ; \quad \eta = 1, \ldots, m .$$

$$\sum_{(bij)} \delta(S_b, S) F_{bij} \left\{ \sum_{k=i}^{j} \delta T_{ijk} \cos(2\pi\eta X_{bk}) \right\} = 0 ,$$
$$S = 1, 2 ; \quad \eta = 1, \ldots, m .$$

$$\sum_{(bij)} \delta(S_b, S) F_{bij} \left\{ \sum_{k=i}^{j} \delta T_{ijk} X_{bk}^{r} \right\} = 0 , \quad S = 1, 2 ; \quad r = 0, \ldots, n .$$

The following iterative method of solution is suggested here:

Substitute an approximate value for $\hat{\beta}_*$ in the first three and last three sets of equations, which are then solved for approximate values of $\hat{\alpha}_{**}$, $\hat{\alpha}'_{**}$, $\hat{m}_{**}$, $\hat{m}'_{**}$, $\hat{m}''_{**}$. This solution is then substituted into the fourth set of equtions, each of which is then solved directly for a value of one of the $\hat{\beta}_D$. The next order approximation of $\hat{\beta}_*$ is then obtained from these latter values by making $\overline{\hat{\beta}} = 1$.

The process is started by substituting $\hat{\beta}_D = 1$, all D , and is continued to convergence.

Within the above iteration, the first three and last three sets of equations are solved by the following process which is also iterative:

Substitute approximate values for the $\hat{\alpha}_S(t)$ into the terms in $\log \hat{\alpha}_S(t)$ and in $1/\hat{\alpha}_S(t)$. Solve the resulting linear equations for the next order approximation of $\hat{\alpha}_{**}$, $\hat{\alpha}'_{**}$, $\hat{m}_{**}$, $\hat{m}'_{**}$, $\hat{m}''_{**}$. The process is continued to convergence.

Example: –

Suppose there are 3 consecutive sampling intervals, each of 1/2 year duration, and 4 zones sampled as in Table 5.

TABLE 5

PATTERN OF SAMPLING IN ILLUSTRATIVE EXAMPLE OF APPLICATION OF MODEL TO A ROCK LOBSTER FISHERY

✓ denotes sampling

	Zone 1	Zone 2	Zone 3	Zone 4
δt_1	✓		✓	✓
δt_2	✓	✓	✓	
δt_3	✓	✓		✓

Each diagonal line in Figure 11 represents age v. time for 8 cohorts (2 sexes in each of 4 zones). The label b of each cohort is shown below this line e.g. 1SD ($S = 1, 2$; $D = 1, 2, 3, 4$).

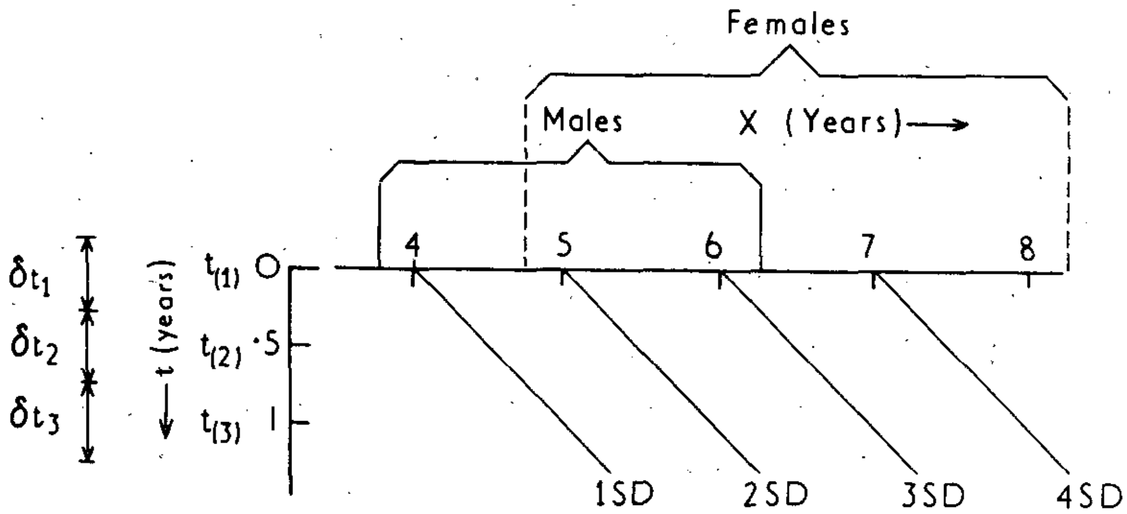

Fig. 11. Diagram of age v. time for the cohorts in the zones of Table 5.

Suppose male cohorts are full sized from $3\frac{3}{4}$ years ($= 4$ years $- \delta t_1 / 2$) onwards but that male cohorts older than $6\frac{1}{4}$ years ($= 6$ years $+ \delta t_3 / 2$) are not sufficiently well represented in the catch to enter the equations for $\hat{q}$, $\hat{M}$. Let the corresponding ages for females be $4\frac{3}{4}$ years ($= 5$ years $- \delta t_1 / 2$) and $8\frac{1}{4}$ years ($= 8$ years $+ \delta t_3 / 2$) respectively. These ages are shown in Figure 11.

Let $\ell = m = n = 1$.

Thus there are 25 equations in the system $\ell_{bij} = L_{bij}$ and 14 unknowns, namely

α_{10}, α'_{11}; α_{20}, α'_{21}; β_1, β_2, β_3, β_4; m'_{11}, m''_{10}, m''_{11}; m'_{21}, m''_{20}, m''_{21} .

With this choice of sampling intervals all sine terms are zero, thus α_{11}, α_{21}, m_{11}, m_{21} do not appear. (If each zone had been sampled during δt_1, δt_2, δt_3 there would be 40 equations and 14 unknowns.)

The least squares method reduces to solving the following equations. (Only the equations $\frac{\partial}{\partial \hat{z}_\ell} \sum_{(bij)} (\hat{\ell}_{bij} - \hat{L}_{bij})^2 = 0$,

$\hat{z}_\ell = \hat{\alpha}_{10}$, $\hat{\beta}_1$, $\hat{m}'_{11}$ are shown. Note $\hat{\alpha}_S(0) = \hat{\alpha}_S(1)$.)

$$\left\{\log\left[\frac{\hat{\alpha}_1(0)}{\hat{\alpha}_1(\cdot5)}\right]+\delta G_{1121}\hat{\alpha}_1(0)\hat{\beta}_1+\delta T_{121}\hat{m}_1(4)+\delta G_{1122}\hat{\alpha}_1(\cdot5)\hat{\beta}_1+\delta T_{122}\hat{m}_1(4\cdot5)-\hat{L}_{111,12}\right\}$$
$$\times\left\{\frac{1}{\hat{\alpha}_1(0)}-\frac{1}{\hat{\alpha}_1(\cdot5)}+\delta G_{1121}\hat{\beta}_1+\delta G_{1122}\hat{\beta}_1\right\}$$

$$+\left\{\log\left[\frac{\hat{\alpha}_1(0)}{\hat{\alpha}_1(\cdot5)}\right]+\delta G_{1121}\hat{\alpha}_1(0)\hat{\beta}_1+\delta T_{121}\hat{m}_1(5)+\delta G_{1122}\hat{\alpha}_1(\cdot5)\hat{\beta}_1+\delta T_{122}\hat{m}_1(5\cdot5)-\hat{L}_{211,12}\right\}$$
$$\times\left\{\frac{1}{\hat{\alpha}_1(0)}-\frac{1}{\hat{\alpha}_1(\cdot5)}+\delta G_{1121}\hat{\beta}_1+\delta G_{1122}\hat{\beta}_1\right\}$$

$$+\left\{\log\left[\frac{\hat{\alpha}_1(\cdot5)}{\hat{\alpha}_1(0)}\right]+\delta G_{1232}\hat{\alpha}_1(\cdot5)\hat{\beta}_1+\delta T_{232}\hat{m}_1(4\cdot5)+\delta G_{1233}\hat{\alpha}_1(0)\hat{\beta}_1+\delta T_{233}\hat{m}_1(5)-\hat{L}_{111,23}\right\}$$
$$\times\left\{\frac{1}{\hat{\alpha}_1(\cdot5)}-\frac{1}{\hat{\alpha}_1(0)}+\delta G_{1232}\hat{\beta}_1+\delta G_{1233}\hat{\beta}_1\right\}$$

$$+\left\{\log\left[\frac{\hat{\alpha}_1(\cdot5)}{\hat{\alpha}_1(0)}\right]+\delta G_{1232}\hat{\alpha}_1(\cdot5)\hat{\beta}_1+\delta T_{232}\hat{m}_1(5\cdot5)+\delta G_{1233}\hat{\alpha}_1(0)\hat{\beta}_1+\delta T_{233}\hat{m}_1(6)-\hat{L}_{211,23}\right\}$$
$$\times\left\{\frac{1}{\hat{\alpha}_1(\cdot5)}-\frac{1}{\hat{\alpha}_1(0)}+\delta G_{1232}\hat{\beta}_1+\delta G_{1233}\hat{\beta}_1\right\}$$

$$+\left\{\log\left[\frac{\hat{\alpha}_1(\cdot5)}{\hat{\alpha}_1(0)}\right]+\delta G_{2232}\hat{\alpha}_1(\cdot5)\hat{\beta}_2+\delta T_{232}\hat{m}_1(4\cdot5)+\delta G_{2233}\hat{\alpha}_1(0)\hat{\beta}_2+\delta T_{233}\hat{m}_1(5)-\hat{L}_{112,23}\right\}$$
$$\times\left\{\frac{1}{\hat{\alpha}_1(\cdot5)}-\frac{1}{\hat{\alpha}_1(0)}+\delta G_{2232}\hat{\beta}_2+\delta G_{2233}\hat{\beta}_2\right\}$$

$$+\left\{\log\left[\frac{\hat{\alpha}_1(\cdot5)}{\hat{\alpha}_1(0)}\right]+\delta G_{2232}\hat{\alpha}_1(\cdot5)\hat{\beta}_2+\delta T_{232}\hat{m}_1(5\cdot5)+\delta G_{2233}\hat{\alpha}_1(0)\hat{\beta}_2+\delta T_{233}\hat{m}_1(6)-\hat{L}_{212,23}\right\}$$
$$\times\left\{\frac{1}{\hat{\alpha}_1(\cdot5)}-\frac{1}{\hat{\alpha}_1(0)}+\delta G_{2232}\hat{\beta}_2+\delta G_{2233}\hat{\beta}_2\right\}$$

$$+\left\{\log\left[\frac{\hat{\alpha}_1(0)}{\hat{\alpha}_1(\cdot5)}\right]+\delta G_{3121}\hat{\alpha}_1(0)\hat{\beta}_3+\delta T_{121}\hat{m}_1(4)+\delta G_{3122}\hat{\alpha}_1(\cdot5)\hat{\beta}_3+\delta T_{122}\hat{m}_1(4\cdot5)-\hat{L}_{113,12}\right\}$$
$$\times\left\{\frac{1}{\hat{\alpha}_1(0)}-\frac{1}{\hat{\alpha}_1(\cdot5)}+\delta G_{3121}\hat{\beta}_3+\delta G_{3122}\hat{\beta}_3\right\}$$

$$+\left\{\log\left[\frac{\hat{\alpha}_1(0)}{\hat{\alpha}_1(\cdot5)}\right]+\delta G_{3121}\hat{\alpha}_1(0)\hat{\beta}_3+\delta T_{121}\hat{m}_1(5)+\delta G_{3122}\hat{\alpha}_1(\cdot5)\hat{\beta}_3+\delta T_{122}\hat{m}_1(5\cdot5)-\hat{L}_{213,12}\right\}$$
$$\times\left\{\frac{1}{\hat{\alpha}_1(0)}-\frac{1}{\hat{\alpha}_1(\cdot5)}+\delta G_{3121}\hat{\beta}_3+\delta G_{3122}\hat{\beta}_3\right\}$$

$$+(\delta G_{4131}\hat{\alpha}_1(0)\hat{\beta}_4+\delta T_{131}\hat{m}_1(4)+\delta G_{4132}\hat{\alpha}_1(\cdot5)\hat{\beta}_4+\delta T_{132}\hat{m}_1(4\cdot5)+\delta G_{4133}\hat{\alpha}_1(0)\hat{\beta}_4$$
$$+\delta T_{133}\hat{m}_1(5)-\hat{L}_{114,13})\times(\delta G_{4131}\hat{\beta}_4+\delta G_{4132}\hat{\beta}_4+\delta G_{4133}\hat{\beta}_4)$$
$$+(\delta G_{4131}\hat{\alpha}_1(0)\hat{\beta}_4+\delta T_{131}\hat{m}_1(5)+\delta G_{4132}\hat{\alpha}_1(\cdot5)\hat{\beta}_4+\delta T_{132}\hat{m}_1(5\cdot5)+\delta G_{4133}\hat{\alpha}_1(0)\hat{\beta}_4$$
$$+\delta T_{133}\hat{m}_1(6)-\hat{L}_{214,13})\times(\delta G_{4131}\hat{\beta}_4+\delta G_{4132}\hat{\beta}_4+\delta G_{4133}\hat{\beta}_4)=0\ .$$

$$\vdots$$

$$\left\{\log\left[\frac{\hat{\alpha}_1(0)}{\hat{\alpha}_1(\cdot 5)}\right]+\delta G_{1121}\hat{\alpha}_1(0)\hat{\beta}_1+\delta T_{121}\hat{m}_1(4)+\delta G_{1122}\hat{\alpha}_1(\cdot 5)\hat{\beta}_1+\delta T_{122}\hat{m}_1(4\cdot 5)-\hat{L}_{111,12}\right\}$$
$$\times\ (\delta G_{1121}\hat{\alpha}_1(0)+\delta G_{1122}\hat{\alpha}_1(\cdot 5))$$

$$+\left\{\log\left[\frac{\hat{\alpha}_1(0)}{\hat{\alpha}_1(\cdot 5)}\right]+\delta G_{1121}\hat{\alpha}_1(0)\hat{\beta}_1+\delta T_{121}\hat{m}_1(5)+\delta G_{1122}\hat{\alpha}_1(\cdot 5)\hat{\beta}_1+\delta T_{122}\hat{m}_1(5\cdot 5)-\hat{L}_{211,12}\right\}$$
$$\times\ (\delta G_{1121}\hat{\alpha}_1(0)+\delta G_{1122}\hat{\alpha}_1(\cdot 5))$$

$$+\left\{\log\left[\frac{\hat{\alpha}_2(0)}{\hat{\alpha}_2(\cdot 5)}\right]+\delta G_{1121}\hat{\alpha}_2(0)\hat{\beta}_1+\delta T_{121}\hat{m}_2(5)+\delta G_{1122}\hat{\alpha}_2(\cdot 5)\hat{\beta}_1+\delta T_{122}\hat{m}_2(5\cdot 5)-\hat{L}_{221,12}\right\}$$
$$\times\ (\delta G_{1121}\hat{\alpha}_2(0)+\delta G_{1122}\hat{\alpha}_2(\cdot 5))$$

$$+\left\{\log\left[\frac{\hat{\alpha}_2(0)}{\hat{\alpha}_2(\cdot 5)}\right]+\delta G_{1121}\hat{\alpha}_2(0)\hat{\beta}_1+\delta T_{121}\hat{m}_2(6)+\delta G_{1122}\hat{\alpha}_2(\cdot 5)\hat{\beta}_1+\delta T_{122}\hat{m}_2(6\cdot 5)-\hat{L}_{321,12}\right\}$$
$$\times\ (\delta G_{1121}\hat{\alpha}_2(0)+\delta G_{1122}\hat{\alpha}_2(\cdot 5))$$

$$+\left\{\log\left[\frac{\hat{\alpha}_2(0)}{\hat{\alpha}_2(\cdot 5)}\right]+\delta G_{1121}\hat{\alpha}_2(0)\hat{\beta}_1+\delta T_{121}\hat{m}_2(7)+\delta G_{1122}\hat{\alpha}_2(\cdot 5)\hat{\beta}_1+\delta T_{122}\hat{m}_2(7\cdot 5)-\hat{L}_{421,12}\right\}$$
$$\times\ (\delta G_{1121}\hat{\alpha}_2(0)+\delta G_{1122}\hat{\alpha}_2(\cdot 5))$$

$$+\left\{\log\left[\frac{\hat{\alpha}_1(\cdot 5)}{\hat{\alpha}_1(0)}\right]+\delta G_{1232}\hat{\alpha}_1(\cdot 5)\hat{\beta}_1+\delta T_{232}\hat{m}_1(4\cdot 5)+\delta G_{1233}\hat{\alpha}_1(0)\hat{\beta}_1+\delta T_{233}\hat{m}_1(5)-\hat{L}_{111,23}\right\}$$
$$\times\ (\delta G_{1232}\hat{\alpha}_1(\cdot 5)+\delta G_{1233}\hat{\alpha}_1(0))$$

$$+\left\{\log\left[\frac{\hat{\alpha}_1(\cdot 5)}{\hat{\alpha}_1(0)}\right]+\delta G_{1232}\hat{\alpha}_1(\cdot 5)\hat{\beta}_1+\delta T_{232}\hat{m}_1(5\cdot 5)+\delta G_{1233}\hat{\alpha}_1(0)\hat{\beta}_1+\delta T_{233}\hat{m}_1(6)-\hat{L}_{211,23}\right\}$$
$$\times\ (\delta G_{1232}\hat{\alpha}_1(\cdot 5)+\delta G_{1233}\hat{\alpha}_1(0))$$

$$+\left\{\log\left[\frac{\hat{\alpha}_2(\cdot 5)}{\hat{\alpha}_2(0)}\right]+\delta G_{1232}\hat{\alpha}_2(\cdot 5)\hat{\beta}_1+\delta T_{232}\hat{m}_2(5\cdot 5)+\delta G_{1233}\hat{\alpha}_2(0)\hat{\beta}_1+\delta T_{233}\hat{m}_2(6)-\hat{L}_{221,23}\right\}$$
$$\times\ (\delta G_{1232}\hat{\alpha}_2(\cdot 5)+\delta G_{1233}\hat{\alpha}_2(0))$$

$$+\left\{\log\left[\frac{\hat{\alpha}_2(\cdot 5)}{\hat{\alpha}_2(0)}\right]+\delta G_{1232}\hat{\alpha}_2(\cdot 5)\hat{\beta}_1+\delta T_{232}\hat{m}_2(6\cdot 5)+\delta G_{1233}\hat{\alpha}_2(0)\hat{\beta}_1+\delta T_{233}\hat{m}_2(7)-\hat{L}_{321,23}\right\}$$
$$\times\ (\delta G_{1232}\hat{\alpha}_2(\cdot 5)+\delta G_{1233}\hat{\alpha}_2(0))$$

$$+\left\{\log\left[\frac{\hat{\alpha}_2(\cdot 5)}{\hat{\alpha}_2(0)}\right]+\delta G_{1232}\hat{\alpha}_2(\cdot 5)\hat{\beta}_1+\delta T_{232}\hat{m}_2(7\cdot 5)+\delta G_{1233}\hat{\alpha}_2(0)\hat{\beta}_1+\delta T_{233}\hat{m}_2(8)-\hat{L}_{421,23}\right\}$$
$$\times\ (\delta G_{1232}\hat{\alpha}_2(\cdot 5)+\delta G_{1233}\hat{\alpha}_2(0)) = 0\ .$$

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$$\left\{\log\left[\frac{\hat{\alpha}_1(0)}{\hat{\alpha}_1(\cdot 5)}\right]+\delta G_{1121}\hat{\alpha}_1(0)\hat{\beta}_1+\delta T_{121}\hat{m}_1(4)+\delta G_{1122}\hat{\alpha}_1(\cdot 5)\hat{\beta}_1+\delta T_{122}\hat{m}_1(4\cdot 5)-\hat{L}_{111,12}\right\}$$
$$\times\ (\delta T_{121}-\delta T_{122})$$

$$+\left\{\log\left[\frac{\hat{\alpha}_1(0)}{\hat{\alpha}_1(\cdot 5)}\right]+\delta G_{1121}\hat{\alpha}_1(0)\hat{\beta}_1+\delta T_{121}\hat{m}_1(5)+\delta G_{1122}\hat{\alpha}_1(\cdot 5)\hat{\beta}_1+\delta T_{122}\hat{m}_1(5\cdot 5)-\hat{L}_{211,12}\right\}$$
$$\times\ (\delta T_{121}-\delta T_{122})$$

$$+\left\{\log\left[\frac{\hat{\alpha}_1(\cdot 5)}{\hat{\alpha}_1(0)}\right]+\delta G_{1232}\hat{\alpha}_1(\cdot 5)\hat{\beta}_1+\delta T_{232}\hat{m}_1(4\cdot 5)+\delta G_{1233}\hat{\alpha}_1(0)\hat{\beta}_1+\delta T_{233}\hat{m}_1(5)-\hat{L}_{111,23}\right\}$$
$$\times\ (-\delta T_{232}+\delta T_{233})$$

$$+\left\{\log\left[\frac{\hat{\alpha}_1(\cdot 5)}{\hat{\alpha}_1(0)}\right]+\delta G_{1232}\hat{\alpha}_1(\cdot 5)\hat{\beta}_1+\delta T_{232}\hat{m}_1(5\cdot 5)+\delta G_{1233}\hat{\alpha}_1(0)\hat{\beta}_1+\delta T_{233}\hat{m}_1(6)-\hat{L}_{211,23}\right\}$$
$$\times\ (-\delta T_{232}+\delta T_{233})$$

$$+\left\{\log\left[\frac{\hat{\alpha}_1(\cdot 5)}{\hat{\alpha}_1(0)}\right]+\delta G_{2232}\hat{\alpha}_1(\cdot 5)\hat{\beta}_2+\delta T_{232}\hat{m}_1(4\cdot 5)+\delta G_{2233}\hat{\alpha}_1(0)\hat{\beta}_2+\delta T_{233}\hat{m}_1(5)-\hat{L}_{112,23}\right\}$$
$$\times\ (-\delta T_{232}+\delta T_{233})$$

$$+\left\{\log\left[\frac{\hat{\alpha}_1(\cdot 5)}{\hat{\alpha}_1(0)}\right]+\delta G_{2232}\hat{\alpha}_1(\cdot 5)\hat{\beta}_2+\delta T_{232}\hat{m}_1(5\cdot 5)+\delta G_{2233}\hat{\alpha}_1(0)\hat{\beta}_2+\delta T_{233}\hat{m}_1(6)-\hat{L}_{212,23}\right\}$$
$$\times\ (-\delta T_{232}+\delta T_{233})$$

$$+\left\{\log\left[\frac{\hat{\alpha}_1(0)}{\hat{\alpha}_1(\cdot 5)}\right]+\delta G_{3121}\hat{\alpha}_1(0)\hat{\beta}_3+\delta T_{121}\hat{m}_1(4)+\delta G_{3122}\hat{\alpha}_1(\cdot 5)\hat{\beta}_3+\delta T_{122}\hat{m}_1(4\cdot 5)-\hat{L}_{113,12}\right\}$$
$$\times\ (\delta T_{121}-\delta T_{122})$$

$$+\left\{\log\left[\frac{\hat{\alpha}_1(0)}{\hat{\alpha}_1(\cdot 5)}\right]+\delta G_{3121}\hat{\alpha}_1(0)\hat{\beta}_3+\delta T_{121}\hat{m}_1(5)+\delta G_{3122}\hat{\alpha}_1(\cdot 5)\hat{\beta}_3+\delta T_{122}\hat{m}_1(5\cdot 5)-\hat{L}_{213,12}\right\}$$
$$\times\ (\delta T_{121}-\delta T_{122})$$

$$+(\delta G_{4131}\hat{\alpha}_1(0)\hat{\beta}_4+\delta T_{131}\hat{m}_1(4)+\delta G_{4132}\hat{\alpha}_1(\cdot 5)\hat{\beta}_4+\delta T_{132}\hat{m}_1(4\cdot 5)+\delta G_{4133}\hat{\alpha}_1(0)\hat{\beta}_4$$
$$+\delta T_{133}\hat{m}_1(5)-\hat{L}_{114,13})\times(\delta T_{131}-\delta T_{132}+\delta T_{133})$$

$$+(\delta G_{4131}\hat{\alpha}_1(0)\hat{\beta}_4+\delta T_{131}\hat{m}_1(5)+\delta G_{4132}\hat{\alpha}_1(\cdot 5)\hat{\beta}_4+\delta T_{132}\hat{m}_1(5\cdot 5)+\delta G_{4133}\hat{\alpha}_1(0)\hat{\beta}_4$$
$$+\delta T_{133}\hat{m}_1(6)-\hat{L}_{214,13})\times(\delta T_{131}-\delta T_{132}+\delta T_{133})=0\ .$$

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XV. ACKNOWLEDGEMENT

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APPENDIX I. ON NOTATION

Flow chart symbols are shown in Figure 12.

An ordered set, or vector, (y_ℓ), where ℓ takes discrete values, is written as y_* .

An ordered array, or matrix, $(y_{\ell m})$, where ℓ and m take discrete values, is written as y_{**} . The vectors $(y_{\ell\cdot})$, $(y_{\cdot m})$, where $y_{\ell\cdot} \equiv \sum_m y_{\ell m}$, $y_{\cdot m} \equiv \sum_\ell y_{\ell m}$, are written as $y_{*\cdot}$, $y_{\cdot *}$ respectively. The vectors $(y_{\ell m_1})$, $(y_{\ell_1 m})$, where ℓ_1 , m_1 are particular values of ℓ , m , are written as y_{*m_1} , $y_{\ell_1 *}$ respectively. $y_{\cdot\cdot} \equiv \sum_{\ell,m} y_{\ell m}$.

The above conventions are easily extended to any ordered array with any number of subscripts.

The determinant of square matrix y_{**} is written $\|y_{**}\|$. The cofactor of $y_{\ell m}$ in $\|y_{**}\|$ is written $\|y_{**}\|_{\ell m}$, and is defined as the determinant resulting when $y_{\ell m}$ is replaced by 1 and the remainder of row ℓ and column m is replaced by zeros.

Table 6 lists symbols with constant meaning throughout the text, and shows the section wherein each of these symbols is defined. Other symbols are defined and redefined as required.

The following statistical operators are used:

BS , bias (expected value minus some defined value)

τ^2 , mean square error (second moment about a defined point)

ρ , correlation coefficient

CV , coefficient of variation (standard deviation / expected value).

The condition is placed after the argument; e.g. $E(c|\Lambda)$ means the expected value of c given the set Λ . If the argument is enclosed by (| , then |R is implied, where R is defined in Section II; e.g. $CV(N|$ means $CV(N|R)$.

An exponent written after a closing bracket is applied before any operator written before the opening bracket e.g. $E(\delta z)^2$ means $E((\delta z)^2)$.

With the exception of the operators σ^2 and τ^2 , an exponent written after an operator is applied after the operation e.g. $E^{\frac{1}{2}}(N|$ means $\{E(N|\}^{\frac{1}{2}}$.

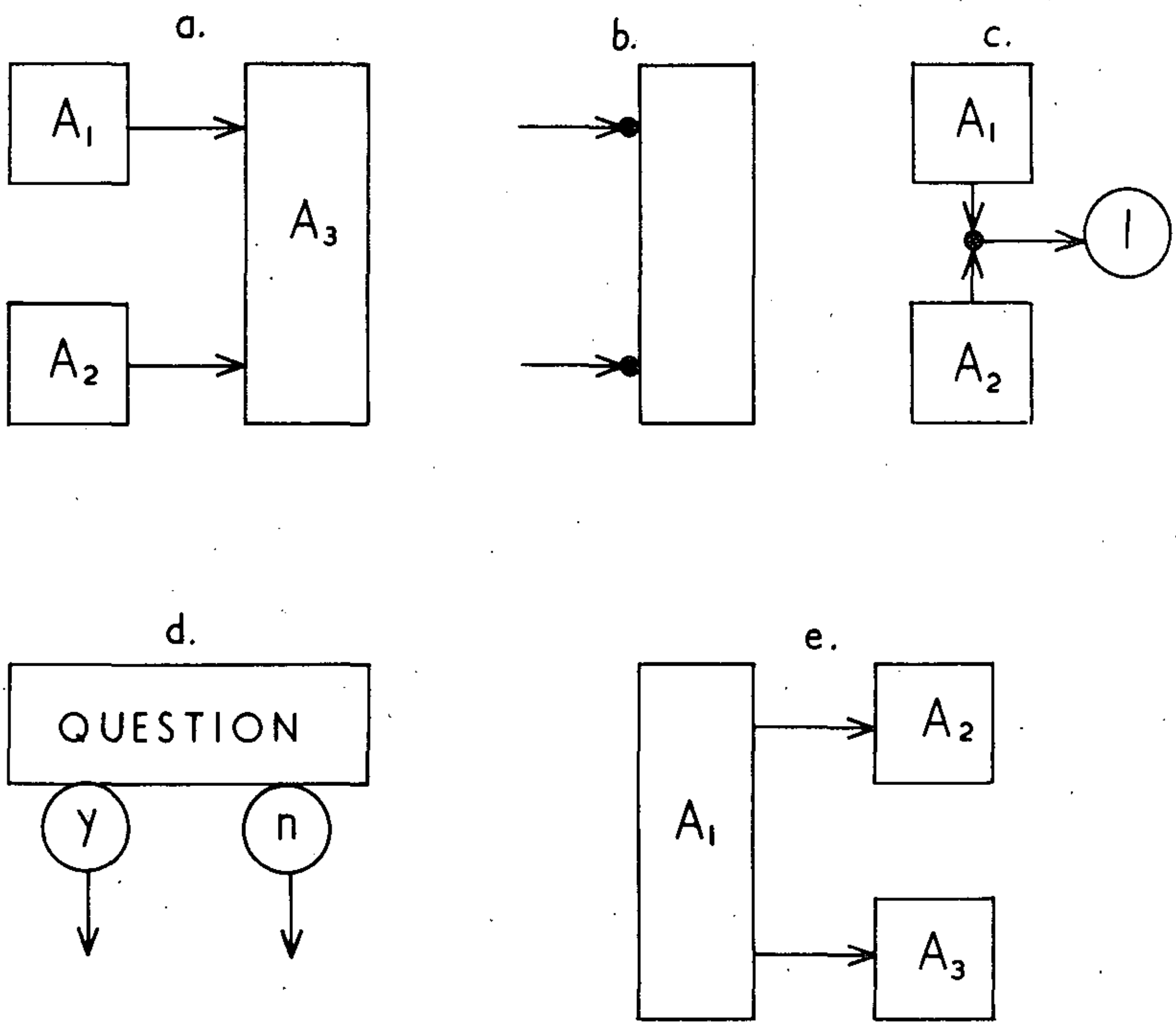

Fig. 12. Flow chart symbols. (a) Perform A_3 after A_1. Perform A_3 after A_2. (b) Different entry points. (c) Perform both A_1 and A_2 before proceeding to 1. (d) Exit at y if answer is yes, at n if answer is no. (e) Perform both A_2 and A_3 after A_1.

TABLE 6

SYMBOLS WITH CONSTANT MEANING THROUGHOUT TEXT

Symbol	defined in Section	Symbol	defined in Section	Symbol	defined in Section
a_i	II	$m_S(X), m_{S\eta}, m'_{S\eta}, m''_{Sr}$	XIII	ζ_i	V(c)
$\bar{a}_i$	IV	N_o, N	II	ζ'_i, ζ''_i	VII(iii)
$a_{\beta i(j)}$	V(b)(i)	N', N''	IV	Λ	IV
b	V	$N_{\alpha\beta}$	V(b)(i)	$\mu_{\alpha\beta}$	V(b)(i)
B_i	V(c)	$Pr, \overline{P}$	II	μ	IV
$\hat{B}_i, \hat{B}'_i, \hat{B}''_i, B'_i, B''_i$	VII(iii)	$p_{\beta i(j)}(\alpha), p_{\alpha\beta}$	V(b)(i)	μ_i	V(c)
BS	Appendix I	$q(h,x,t), q_i, q$	II	ν	V(c)
c	IV	$q', q'', \hat{q}, \hat{q}', \hat{q}''$	IV	ν', ν''	VII(iii)
$c_{\alpha\beta}$	V	Q', Q''	IV(Fig.1)	ξ, ξ', ξ''	IV
$c_{\alpha\beta}(J), \hat{c}_{\alpha\beta}$	V(b)(ii)	$q_{bi}, \hat{q}_{bi}$	XI	$\hat{\xi}$	IX
c_i	V(c)	R	II	$\hat{\xi}'; \hat{\xi}''$	VIII(b)
CV	Appendix I	S(t)	II	ρ	Appendix I
dt, dg, dP_i	II	s,s',s''	IV	τ	Appendix I
D,D(b)	XI	S, S_β	V	$\phi_{\beta\alpha}$	V(b)(i)
F	II	t, t_o	II	$\chi_{\beta\alpha}$	V(b)(i)
g'	II	t',t''	IV	$\psi_{\beta\alpha}$	V(b)(i)
$h, h_i(x)$	II	$t_{\beta(j)}$	V(b)(i)	$\frown$	V(c)
H_α	V	$t_{(i)}$	XI		
$\hat{K}_i$	VII(iii)	X, X_r, x	II		
$L, \hat{L}$	IV	$X_\beta, X_{\beta(j)}$	V		
$\ell(\hat{K}_*, B'_*, B''_*)$	VII(iii)	Z	II		
$\hat{\ell}_{bij}, L_{bij}, \hat{L}_{bij}$	XI	$z_i, \hat{z}_i$	XI		
$M(h,x,t), M_i, M$	II	$\alpha_{**}, \alpha'_{S\eta}, \alpha_S(t)$	XIII		
$\hat{M}$	IV	$\beta_i(j), \beta_i$	V(b)(i)		
$M_{bi}, \hat{M}_{bi}$	XI	β_D	XIII		
		Γ	V		
		$\gamma_{\alpha\beta}$	V(b)(i)		
		$\delta g, \delta g', \delta g'', \delta t', \delta t''$,	IV		
		δt	IV,XI		
		$\delta G_{D(b)ijk}, \delta T_{ijk}, \delta t_i$	XI		

APPENDIX II. RATIO OF VARIATES

Let v_1, v_2 be variates and write $BS(v_2/v_1) \equiv E(v_2/v_1) - E(v_2)/E(v_1)$, $\varepsilon \equiv |CV(v_1)| + |CV(v_2)|$. Then

(1) $|BS(v_2/v_1)|/\sigma(v_2/v_1) \le |CV(v_1)|$.

(2) $BS(v_2/v_1) = \frac{E(v_2)}{E(v_1)}[CV^2(v_1) - \rho(v_1,v_2)CV(v_1)CV(v_2) + O(\varepsilon^3)]$.

(3) $E(v_2/v_1 - E(v_2)/E(v_1))^2 =$
$\frac{E^2(v_2)}{E^2(v_1)}[CV^2(v_1) + CV^2(v_2) - 2\rho(v_1,v_2)CV(v_1)CV(v_2) + O(\varepsilon^3)]$.

(4) $CV^2(v_2/v_1) = CV^2(v_1) + CV^2(v_2) - 2\rho(v_1,v_2)CV(v_1)CV(v_2) + O(\varepsilon^3)$.

Proof: –

(1), (2), (3) constitute Exercise 107 of Raj (1968).

(4) follows from (2) and (3).

Q.E.D.

APPENDIX III. UPPER BOUND TO VARIANCE OF ELEMENT OF INVERSE MATRIX

Let V_{**} be a non-singular n×n matrix of variates. Then

$$\sigma[(V_{**}^{-1})_{ij}] \le E^{\frac{1}{2}}\{(V_{**}^{-1})_{ij} - (E^{-1}[V_{**}])_{ij}\}^2$$

$$\le \sum_{\ell,m=1}^{n} \left| \frac{\varepsilon_{\ell j}\varepsilon_{mi} \| \, \|E(V_{**})\|_{ji} \|_{\ell m}}{\|E(V_{**})\|} - \frac{\|E(V_{**})\|_{ji} \, \|E(V_{**})\|_{\ell m}}{\|E(V_{**})\|^2} \right| \sigma(V_{\ell m}) ,$$

where $\varepsilon_{\alpha\beta} = 0,1$ if $\alpha = \beta$, $\alpha \ne \beta$.

Proof: –

$(V_{**}^{-1})_{ij} = \|V_{**}\|_{ji} / \|V_{**}\|$ and is a function of $(V_{11}, \ldots, V_{nn})$. Now $\frac{\partial}{\partial V_{\ell m}} \|V_{**}\| = \|V_{**}\|_{\ell m}$. So

$\frac{\partial}{\partial V_{\ell m}} \|V_{**}\|_{ji} = \| \, \|V_{**}\|_{ji} \|_{\ell m}$, 0 if $\ell \ne j$ and $m \ne i$, $\ell = j$ or $m = i$. Hence

$$\frac{\partial}{\partial V_{\ell m}} (V_{**}^{-1})_{ij} = \frac{1}{\|V_{**}\|} \frac{\partial \|V_{**}\|_{ji}}{\partial V_{\ell m}} - \frac{\|V_{**}\|_{ji}}{\|V_{**}\|^2} \frac{\partial \|V_{**}\|}{\partial V_{\ell m}}$$

$$= \frac{\varepsilon_{\ell j}\varepsilon_{mi} \| \, \|V_{**}\|_{ji} \|_{\ell m}}{\|V_{**}\|} - \frac{\|V_{**}\|_{ji} \|V_{**}\|_{\ell m}}{\|V_{**}\|^2} .$$ By the

corollary of Appendix IV, the result follows.

Q.E.D.

The following program, subroutine SIGIN, calculates the upper bound for $n \geq 3$. Input is n, $E(v_{**})$ and $\sigma(v_{**})$, denoted in the subroutine by N, X, S respectively, and the matrix of upper bounds of the $\sigma[(v_{**}^{-1})_{ij}]$, denoted by T, is returned. SIGIN calculates term ℓ, m of T(i,j)

$$\text{as} \quad \left| \frac{\varepsilon \, || R_j, R_\ell, C_i, C_m ||}{|| E(v_{**}) ||} - \frac{|| R_j, C_i || \, || R_\ell, C_m ||}{|| E(v_{**}) ||^2} \right| \sigma(v_{\ell m}) ,$$

where $\varepsilon = 0$ if $\ell = j$ or $m = i$,

$\quad\quad 1$ if $(\ell > j$ and $m > i)$ or $(\ell < j$ and $m < i)$,

$\quad -1$ otherwise

and $|| R_\alpha, R_\beta, \ldots, C_A, C_B, \ldots ||$ means the determinant resulting when rows $\alpha, \beta, \ldots$, and columns $A, B, \ldots$ are eliminated from $|| E(v_{**}) ||$.

(Language: CONTROL DATA 6600 Extended FORTRAN)

```
      SUBROUTINESIGIN(N,X,S,T)
      DIMENSIONX(Ø4,Ø4),AX(Ø4,Ø4),S(Ø4,Ø4),T(Ø4,Ø4),Y1(Ø3,Ø3),Y2(Ø3,Ø3),
     1Y3(Ø2,Ø2)
      N1=N-1$N2=N-2
      DO4I=1,N$DO4J=1,N
    4 AX(I,J)=X(I,J)$CALLXCDETRX(N,N,AX,DAX)$DO1I=1,N$DO1J=1,N
      DO2IGA=1,N1 $DO2IDE=1,N1 $MU=IGA$IF(IGA.GE.J)MU=IGA+1
      NU=IDE$IF(IDE.GE.I)NU=IDE+1
    2 Y2(IGA,IDE)=X(MU,NU)$CALLXCDETRX(N-1,N-1,Y2,DY2)$T(I,J)=Ø$DO1L=1,N
      DO1M=1,N$EPS=-1$IF(L.EQ.J.OR.M.EQ.I)EPS=Ø
      IF((L-J)*(M-I).GT.Ø )EPS=1$DO3IGA=1,N1 $DO3IDE=1,N1 $MU=IGA
      IF(IGA.GE.L)MU=IGA+1$NU=IDE$IF(IDE.GE.M)NU=IDE+1
    3 Y1(IGA,IDE)=X(MU,NU)$CALLXCDETRX(N-1,N-1,Y1,DY1)$R1=AMINØ(J,L)
      C1=AMINØ(I,M)$R2=AMAXØ(J,L)$C2=AMAXØ(I,M)$DO5IGA=1,N2
      DO5IDE=1,N2 $MU=IGA+1$IF(IGA.LT.R1)MU=IGA$IF(IGA.GE.R2-1)MU=IGA+2
      NU=IDE+1$IF(IDE.LT.C1)NU=IDE$IF(IDE.GE.C2-1)NU=IDE+2
    5 Y3(IGA,IDE)=X(MU,NU)$IF(N2.NE.1)CALLXCDETRX(N-2,N-2,Y3,DY3)
      IF(N2.EQ.1)DY3=Y3(1,1)
    1 T(I,J)=T(I,J)+ABS(S(L,M)*(EPS*DY3/DAX-DY2*DY1/(DAX)**2))
      END
```

APPENDIX IV. UPPER BOUND TO MEAN SQUARE ERROR OF A FUNCTION OF VARIATES

Let $z(v_*)$ be a function of the variates $v_1, v_2, \ldots$ and let r_* be any real vector. Let $\delta v_* \equiv v_* - r_*$, $\delta z \equiv z(v_*) - z(r_*)$, $\delta_i z \equiv z(r_1, \ldots, r_{i-1}, v_i, r_{i+1}, r_{i+2}, \ldots) - z(r_*)$, $\tau^2(v_i) \equiv E(\delta v_i)^2$, $\tau^2(z) \equiv E(\delta z)^2$, $\tau_i^2(z) \equiv E(\delta_i z)^2$. Then, if $z(v_*)$ can be expanded as a Taylor series about r_* ,

$$\tau(z) \le \sum \tau_i(z) + O[\Sigma\tau_i(z)]^2 = \sum \left| \frac{\partial z(r_*)}{\partial v_i} \right| \tau(v_i) + O[\Sigma\tau(v_i)]^2 .$$

Proof: –

$$\delta z = [z(v_*) - z(r_1, v_2, v_3, \ldots)] + [z(r_1, v_2, v_3, \ldots) - z(r_1, r_2, v_3, \ldots)] + [z(r_1, r_2, v_3, \ldots) - z(r_1, r_2, r_3, v_4, \ldots)] + \ldots + [\ldots - z(r_*)]$$

$= \sum \delta_i z + O\left(\sum |\delta v_i|\right)^2$. On squaring, taking expected values and applying Appendix V, the first relation is proved.

$\delta_i z = \frac{\partial z(r_*)}{\partial v_i} \delta v_i + O(\delta v_i)^2$. On squaring and taking expected expected values, the second relation is proved.

Q.E.D.

Corollary: –

$$\sigma(z[v_*]) \le E^{\frac{1}{2}}\{z(v_*) - z(E[v_*])\}^2 \le \sum \left| \frac{\partial z(E[v_*])}{\partial v_i} \right| \sigma(v_i) + O[\Sigma\sigma(v_i)]^2 .$$

Proof: – Substitute $E(v_*)$ for r_* and note $\sigma(z) \le \tau(z)$.

Q.E.D.

APPENDIX V. LOWER BOUND TO PRODUCT OF MEAN SQUARE ERRORS OF TWO VARIATES

Let $\tau^2(v_1) \equiv E(v_1 - r_1)^2$; $\tau^2(v_2) \equiv E(v_2 - r_2)^2$ where v_1 , v_2 are variates and r_1 , r_2 are any two real numbers. Then
$\tau(v_1)\tau(v_2) \geq |E(v_1 - r_1)(v_2 - r_2)|$.

Proof: –

$$|E(v_1 - r_1)(v_2 - r_2)|$$
$$= |E(v_1v_2) - E(v_1)r_2 - r_1E(v_2) + r_1r_2|$$
$$= |E(v_1v_2) - E(v_1)E(v_2) + E(v_1)E(v_2) - E(v_1)r_2 - r_1E(v_2) + r_1r_2|$$
$$= |\mathrm{cov}(v_1,v_2) + BS(v_1)BS(v_2)| \quad (\text{where } BS(v_1) \equiv E(v_1) - r_1 , \; BS(v_2) \equiv E(v_2) - r_2)$$
$$= |\rho(v_1,v_2)\sigma(v_1)\sigma(v_2) + BS(v_1)BS(v_2)|$$
$$\leq \sigma(v_1)\sigma(v_2) + |BS(v_1)|\,|BS(v_2)| \; .$$

Now use the Cauchy - Schwarz inequality, namely:
For real numbers, $(k_1\ell_1 + k_2\ell_2 + \ldots + k_m\ell_m)^2 \leq (k_1^2 + \ldots + k_m^2)(\ell_1^2 + \ldots + \ell_m^2)$.

$$\therefore (\sigma(v_1)\sigma(v_2) + |BS(v_1)|\,|BS(v_2)|)^2 \leq (\sigma^2(v_1) + BS^2(v_1))(\sigma^2(v_2) + BS^2(v_2))$$
$$= \tau^2(v_1)\tau^2(v_2) \; .$$

Q.E.D.

APPENDIX VI. METHOD FOR NUMERICAL EVALUATION OF

$$I = \int f(v_*)z(v_*)dv_* \text{ WHERE } \int f(v_*)dv_* = 1 ,$$

$$\text{OR } I = \sum_{v_*} f(v_*)z(v_*) \text{ WHERE } \sum_{v_*} f(v_*) = 1$$

Using a computer, select n values of v_* according to the probability law $f(v_*)$. Let v_{*i} be the value selected on the i^{th} occasion. Let

$$\bar{z} \equiv \frac{1}{n}\sum_{i=1}^{n} z(v_{*i}) , \quad s^2_{\bar{z}} \equiv \frac{1}{n-1}\sum_{i=1}^{n}(z(v_{*i}) - \bar{z})^2$$

. Then $E(\bar{z}) = I$, $E(s^2_{\bar{z}}/n) = \sigma^2(\bar{z})$. So continue the process till $(s^2_{\bar{z}}/n)^{\frac{1}{2}} / \bar{z}$ is sufficiently small.

APPENDIX VII. PROGRAM FOR INVESTIGATING ESTIMATOR ζ_i DEFINED IN SECTION VI

The symbols in the following program are defined in Table 7.
(Language: CONTROL DATA 6600 Extended FORTRAN)

```
      PROGRAM WS(INPUT,TAPE1=INPUT,OUTPUT)
      DIMENSIONAN(5Ø),CI(5Ø),S2(5Ø)
      DIMENSIONCON(1ØØ),IND(5Ø)
      TN=TCI=K=Ø
    1 K=K+1
      READ(1,2)AN(K),CI(K)
      IF(EOF(1).NE.Ø)GOTO22
    2 FORMAT(2F4.Ø)
      TN=TN+AN(K)
      TCI=TCI+CI(K)
      S2(K)=CI(K)*(1-CI(K)/AN(K))/(AN(K)-1)
      GOTO1
   22 K=K-1
      S2W=S2B=Ø
      AK=K
      CBAR=TCI/AK
      DO3J=1,K
      S2B=S2B+(CI(J)-CBAR)**2
    3 S2W=S2W+S2(J)
      AK1=K-1
      S2B=S2B/AK1
      S2W=S2W/AK
      CVS2J=Ø
      DO4J=1,K
    4 CVS2J=CVS2J+(S2(J)-S2W)**2
      CVS2J=SQRT(CVS2J/AK)/S2W
      PRINT5,CVS2J
    5 FORMAT(1H1,*CVS2J=*,E11.4)
      N=TN
      PRINT13,K,N
   13 FORMAT(1X,*K=*,I3,5X
     1                 *N=*,I5/2X,*LK*,5X,*FK*,5X,*FN*,7X,*BS*,7X,*SS*)
   19 READ(1,2Ø)LK,NFN
   2Ø FORMAT(2I3)
      IF(EOF(1).NE.Ø)GOTO21
      FK=LK/AK
      E=Ø
      DO7I1=1,1ØØ
      TCID=TND=Ø
      DO14J=1,K
```

```
14 IND(J)=Ø
   DO8I2=1,LK
15 J=(K-1.E-6)*RANF(DUM)+1
   IF(IND(J).EQ.1)GOTO15
   IND(J)=1
   TCID=TCID+CI(J)
 8 TND=TND+AN(J)
 7 E=E+TCID/TND
   B=TN*E/1ØØ.-TCI
   DO6I=1,NFN
   READ18,FN
18 FORMAT(F5.2)
   E1=E2=Ø
   DO9I1=1,1ØØ
   TNS2J=TCID=TND=Ø
   DO16J=1,K
16 IND(J)=Ø
   DO1ØI2=1,LK
17 J=(K-1.E-6)*RANF(DUM)+1
   IF(IND(J).EQ.1)GOTO17
   IND(J)=1
   TCID=TCID+CI(J)
   TND=TND+AN(J)
1Ø TNS2J=TNS2J+AN(J)*S2(J)
   CON(I1)=TCID/TND
   E1=E1+CON(I1)
 9 E2=E2+TNS2J/TND**2
   E1=E1/1ØØ.
   V=Ø
   DO11I1=1,1ØØ
11 V=V+(CON(I1)-E1)**2
   SIG=SQRT(TN**2*V/99. +TN**2*(FK/FN-1)*E2/1ØØ.)
   BS=B/SIG
   SIGS=SQRT(K*S2B*(1/FK-1)+TN*S2W*(1/FN-1/FK))
   SS=SIGS/SIG
 6 PRINT12,LK,FK,FN,BS,SS
12 FORMAT(1X,I3,2F7.4,2F9.4)
   GOTO19
21 CONTINUE
   END
```

TABLE 7

DEFINITION OF SYMBOLS USED IN PROGRAM OF APPENDIX VII

Symbol	Definition	Symbol	Definition
AN(J)	$C_{..}(J)$	FK	f_k
CI(J)	$C_{i[\cdot,s_b]}(J)$	FN	$f_{\bar{n}}$
S2(J)	S_J^2	BS	$BS(\zeta_i\|s)\ /\ \sigma(\zeta_i\|s)$
TN,N	$C_{..}$	SS	$\sigma(\zeta_i^*\|s)\ /\ \sigma(\zeta_i\|s)$
TCI	C_i	TCID	$C'_{i[\cdot,s_b]}$
K,AK	K	TND	$C'_{..}$
S2B	$S^2_{(B)}$	B	$BS(\zeta_i\|s)$
S2W	$S^2_{(W)}$	SIG	$\sigma(\zeta_i\|s)$
LK	k	SIGS	$\sigma(\zeta_i^*\|s)$
CVS2J	$\left\{\frac{1}{K}\sum_{J=1}^{K}(S_J^2 - S^2_{(W)})^2\right\}^{\frac{1}{2}} / S^2_{(W)}$		

APPENDIX VIII. UPPER BOUND TO MEAN SQUARE ERROR OF SUM OF TWO VARIATES

Let $\tau^2(v_1) \equiv E(v_1 - r_1)^2$, $\tau^2(v_2) \equiv E(v_2 - r_2)^2$, $\tau^2(v_1 + v_2) \equiv E(v_1 + v_2 - [r_1 + r_2])^2$ where v_1 , v_2 are variates and r_1 , r_2 are any two real numbers. Then $\tau(v_1 + v_2) \le \tau(v_1) + \tau(v_2)$.

Proof: — $\tau^2(v_1 + v_2) = E([v_1 - r_1] + [v_2 - r_2])^2$

$\le \tau^2(v_1) + \tau^2(v_2) + 2|E(v_1 - r_1)(v_2 - r_2)|$

$\le \tau^2(v_1) + \tau^2(v_2) + 2\tau(v_1)\tau(v_2)$ by Appendix V.

Q.E.D.

APPENDIX IX. CONFIDENCE INTERVALS USING MEAN SQUARE ERROR

Let $\tau^2(v) \equiv E(v - r)^2$ where v is a variate and r is any real number, and let $\eta > 0$. Then $Pr\{|v - r| < (\eta^2 + 1)^{\frac{1}{2}}\tau(v)\} \ge Pr\{|v - E(v)| < \eta\sigma(v)\}$.

Proof: — $\tau^2(v) = \sigma^2(v) + BS^2(v)$ where $BS(v) \equiv E(v) - r$. Also $E(v) - \eta\sigma(v) = r - (\eta\sigma(v) - BS(v))$ and $E(v) + \eta\sigma(v) = r + \eta\sigma(v) + BS(v)$. Now use the Cauchy - Schwarz inequality (Appendix V): Let $k_1 = \eta$, $\ell_1 = \sigma(v)$, $k_2 = \pm 1$, $\ell_2 = BS(v)$. Then $|\eta\sigma(v) \pm BS(v)| \le (\eta^2 + 1)^{\frac{1}{2}}\tau(v)$.

Q.E.D.